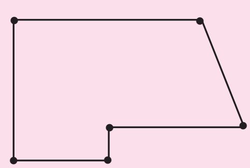
CHAPTER

7

GEOMETRIC FIGURES, AREAS, AND VOLUMES

A carpenter is building a deck on the back of a house. As he works, he follows a plan that he made in the form of a drawing or blueprint. His blueprint is a model of the deck that he is building.

He begins by driving stakes into the ground to locate corners of the deck. Between each pair of stakes, he stretches a cord to indicate the edges of the deck. On the blueprint, the stakes are shown as points and the cords as segments of straight lines as shown in the sketch.



At each corner, the edges of the deck meet at an angle that can be classified according to its size.

Geometry combines points, lines, and planes to model the world in which we live. An understanding of geometry enables us to understand relationships involving the sizes of physical objects and the magnitude and direction of the forces that interact in daily life.

In this chapter, you will review some of the information that you already know to describe angles and apply this information to learn more about geometry.

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7-1 POINTS, LINES, AND PLANES

Undefined Terms

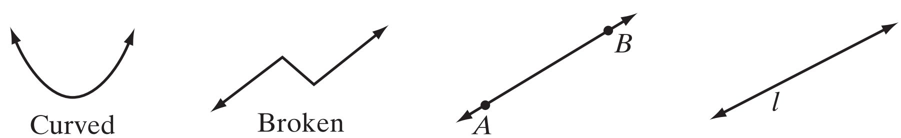
We ordinarily define a word by using a simpler term. The simpler term can be defined by using one or more still simpler terms. But this process cannot go on endlessly; there comes a time when the definition must use a term whose meaning is *assumed* to be clear to all people. Because the meaning is accepted without definition, such a term is called an **undefined term**.

In geometry, we use such ideas as *point*, *line*, and *plane*. Since we cannot give satisfactory definitions of these words by using simpler defined words, we will consider them to be undefined terms.

Although *point*, *line*, and *plane* are undefined words, we must have a clear understanding of what they mean. Knowing the properties and characteristics they possess helps us to achieve this understanding.

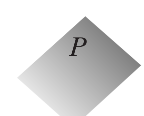
A **point** indicates a place or position. It has no length, width, or thickness. A point is usually indicated by a dot and named with a capital letter. For example, point A is shown on the left.

A **line** is a set of points. The set of points may form a curved line, a broken line, or a straight line. A **straight line** is a line that is suggested by a stretched string but that extends without end in both directions.

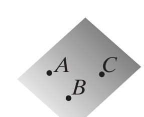


Curved line   Broken line   Straight line

Unless otherwise stated, in this discussion the term *line* will mean *straight* line. A line is named by any two points on the line. For example, the straight line shown above is line AB or line BA, usually written as $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$. A line can also be named by one lowercase letter, for example, line l shown above. The arrows remind us that the line continues beyond what is drawn in the diagram.



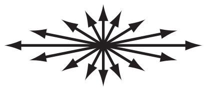
A **plane** is a set of points suggested by a flat surface. A plane extends endlessly in all directions. A plane may be named by a single letter, as plane P shown on the right.



A plane can also be named by three points of the plane, as plane ABC in the diagram on the right.

Facts About Straight Lines

A statement that is accepted as true without proof is called an **axiom** or a **postulate**. If we examine the three accompanying figures pictured on the next page, we see that it is reasonable to accept the following three statements as postulates:



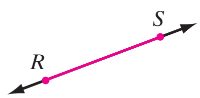
1. In a plane, an infinite number of straight lines can be drawn through a given point.

2. One and only one straight line can be drawn that contains two given points. (Two points determine a straight line.)

3. In a plane, two different nonparallel straight lines will intersect at only one point.

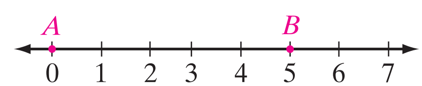
Line Segments

The undefined terms, point, line, and plane, are used to define other geometric terms. A **line segment** or **segment** is a part of a line consisting of two points called **endpoints** and all points on the line between these endpoints.



At the left is pictured a line segment whose endpoints are points R and S. We use these endpoints to name this segment as segment RS, which may be written as $\overline{RS}$ or $\overline{SR}$.

Recall that the **measure of a line segment** or the **length of a line segment** is the distance between its endpoints. We use a number line to associate a number with each endpoint. Since the coordinate of A is 0 and the coordinate of B is 5, the length of $\overline{AB}$ is $5 - 0$ or $AB = 5$.

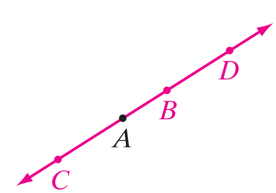


Note: The segment is written as $\overline{AB}$, with a bar over the letters that name the endpoints. The length of the segment is written as AB, with no bar over the letters that name the endpoints.

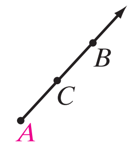
Half-Lines and Rays

When we choose any point A on a line, the two sets of points that lie on opposite sides of A are called **half-lines**. Note that point A is not part of the half-line. In the diagram below, point A separates $\overleftrightarrow{CD}$ into two half-lines.

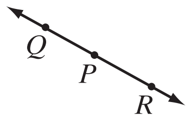
The two points B and D belong to the same half-line since A is not a point of $\overline{BD}$. Points B and C, however, do not belong to the same half-line since A is a point of $\overline{CB}$. All points in the same half-line are said to be on the same side of A. We often talk about rays of sunlight, that is, the sun and the path that the sunlight travels to the earth. The ray of sunlight can be thought of as a point and a half-line.



In geometry, a **ray** is a part of a line that consists of a point on the line, called an **endpoint**, and all the points on one side of the endpoint. To name a ray we use two capital letters and an arrow with one arrowhead. The first letter must be the letter that names the endpoint. The second letter is the name of any other point on the ray.



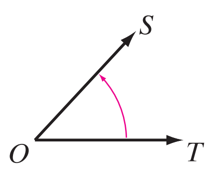
The figure on the right shows ray AB, which is written as $\overrightarrow{AB}$. This ray could also be called ray AC, written as $\overrightarrow{AC}$.



Two rays are called **opposite rays** if they are rays of the same line that have a common endpoint but no other points in common. In the diagram on the left, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are opposite rays.

Angles

An **angle** is a set of points that is the union of two rays having the same endpoint. The common endpoint of the two rays is the **vertex** of the angle. The two rays forming the angle are also called the **sides** of the angle.



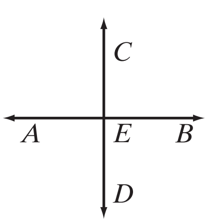
In the diagram on the left, we can think of $\angle TOS$ as having been formed by rotating $\overrightarrow{OT}$ in a counterclockwise direction about O to the position $\overrightarrow{OS}$. The union of the two rays, $\overrightarrow{OT}$ and $\overrightarrow{OS}$, that have the common endpoint O, is $\angle TOS$. Note that when three letters are used to name an angle, the letter that names the vertex is always in the middle. Since $\overrightarrow{OT}$ and $\overrightarrow{OS}$ are the only rays in the diagram that have the common endpoint O, the angle could also have been called $\angle O$.

Measuring Angles

To measure an angle means to determine the number of units of measure it contains. A common standard unit of measure of an angle is the **degree**; 1 degree is written as $1°$. A degree is $\frac{1}{360}$ of the sum of all of the distinct angles about a point. In other words, if we think of an angle as having been formed by rotating a ray around its endpoint, then rotating a ray consecutively 360 times in 1-degree increments will result in one complete rotation.

Types of Angles

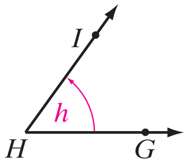
Angles are classified according to their measures.



A **right angle** is an angle whose measure is 90°. In the diagram on the left, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E so that the four angles formed are right angles. The measure of each angle is 90° and the sum of the angles about point E is 360°. We can say that

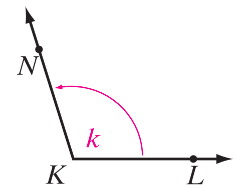
$$m\angle AEC = m\angle CEB = m\angle BED = m\angle DEA = 90.$$

Note that the symbol $m\angle AEC$ is read "the measure of angle AEC." In this book, the angle measure will always be given in degrees and the degree symbol will be omitted when using the symbol "m" to designate measure.



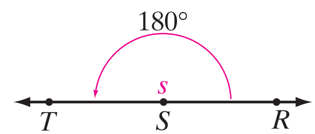
Acute angle

An **acute angle** is an angle whose measure is greater than 0° and less than 90°, that is, the measure of an acute angle is between 0° and 90°. Angle *GHI*, which can also be called $\angle h$, is an acute angle ($0 < m\angle h < 90$).



Obtuse angle

An **obtuse angle** is an angle whose measure is greater than 90° but less than 180°; that is, its measure is between 90° and 180°. Angle *LKN* is an obtuse angle where $90 < m\angle LKN < 180$.



Straight angle

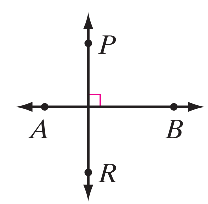
A **straight angle** is an angle whose measure is 180°. Angle *RST* is a straight angle where $m\angle RST = 180$. A straight angle is the union of two opposite rays and forms a straight line.

Here are three important facts about angles:

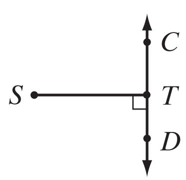
1. The measure of an angle depends only on the amount of rotation, not on the pictured lengths of the rays forming the angle.
2. Since every right angle measures 90°, all right angles are equal in measure.
3. Since every straight angle measures 180°, all straight angles are equal in measure.

Perpendicularity

Two lines are **perpendicular** if and only if the two lines intersect to form right angles. The symbol for perpendicularity is $\perp$.



In the diagram, $\overleftrightarrow{PR}$ is perpendicular to $\overleftrightarrow{AB}$, symbolized as $\overleftrightarrow{PR} \perp \overleftrightarrow{AB}$. The symbol ⏋ is used in a diagram to indicate that a right angle exists where the perpendicular lines intersect.



Segments of perpendicular lines that contain the point of intersection of the lines are also perpendicular. In the diagram on the left, $\overline{ST} \perp \overleftrightarrow{CD}$.

EXERCISES

Writing About Mathematics

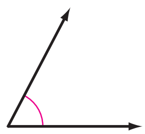
1. Explain how the symbols AB, $\overline{AB}$, and $\overleftrightarrow{AB}$ differ in meaning.

2. Explain the difference between a half-line and a ray.

Developing Skills

In 3–8, tell whether each angle appears to be acute, right obtuse, or straight.

3.



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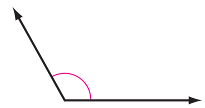

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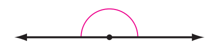

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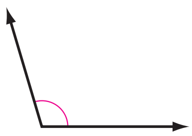

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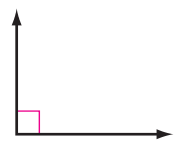


8.

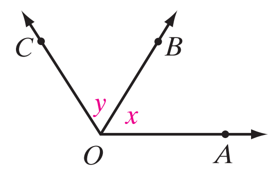
9. For the figure on the right:

a. Name $\angle x$ by using three capital letters.

b. Give the shorter name for $\angle COB$.

c. Name one acute angle.

d. Name one obtuse angle.



Applying Skills

In 10–13, find the number of degrees in the angle formed by the hands of a clock at each given time.

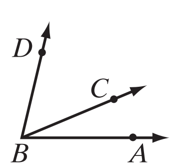
10. 1 P.M. **11.** 4 P.M. **12.** 6 P.M. **13.** 5:30 P.M.

14. At what time do the hands of the clock form an angle of 0°?

15. At what times do the hands of a clock form a right angle?

7-2 PAIRS OF ANGLES

Adjacent Angles



Adjacent angles are two angles in the same plane that have a common vertex and a common side but do not have any interior points in common. In the figure on the left, $\angle ABC$ and $\angle CBD$ are adjacent angles.

Complementary Angles

Two angles are **complementary angles** if and only if the sum of their measures is 90°. Each angle is the *complement* of the other.

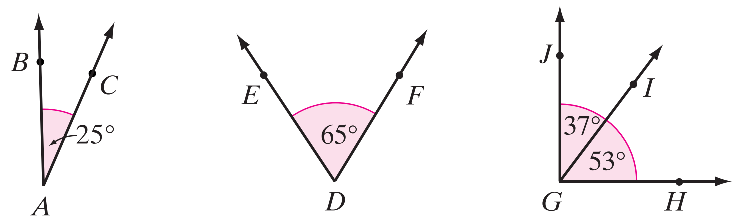
In the figures shown below, because

$$m\angle CAB + m\angle FDE = 25 + 65 = 90,$$

$\angle CAB$ and $\angle FDE$ are complementary angles. Also, because

$$m\angle HGI + m\angle IGJ = 53 + 37 = 90,$$

$\angle HGI$ and $\angle IGJ$ are complementary angles.



If the measure of an angle is 50°, the measure of its complement is $(90 - 50)°$, or 40°. In general,

► **If the measure of an angle is $x°$, the measure of its complement is $(90 - x)°$.**

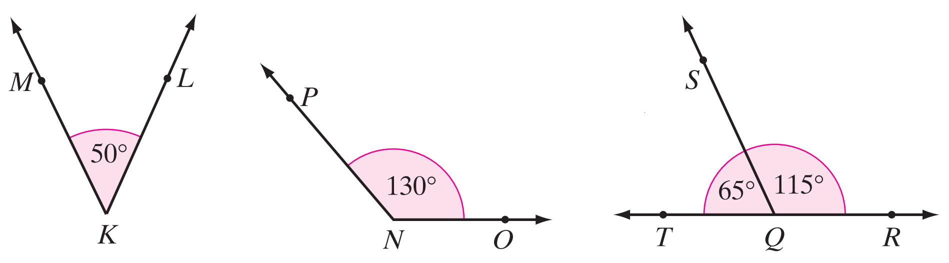
Supplementary Angles

Two angles are **supplementary angles** if and only if the sum of their measures is 180°. Each angle is the *supplement* of the other. As shown in the figures below, because

$$m\angle LKM + m\angle ONP = 50 + 130 = 180,$$

$\angle LKM$ and $\angle ONP$ are supplementary angles. Also, $\angle RQS$ and $\angle SQT$ are supplementary angles because

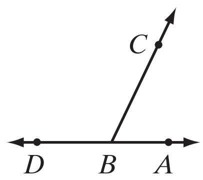
$$m\angle RQS + m\angle SQT = 115 + 65 = 180.$$



If the measure of an angle is 70°, the measure of its supplement is $(180 - 70)°$, or 110°. In general,

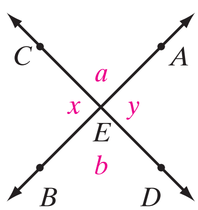
► **If the measure of an angle is $x°$, the measure of its supplement is $(180 - x)°$.**

Linear Pair



Through two points, D and A, draw a line. Choose any point B on $\overline{DA}$ and any point C not on $\overleftrightarrow{DA}$. Draw $\overrightarrow{BC}$. The adjacent angles formed, $\angle DBC$ and $\angle CBA$, are called a linear pair. A **linear pair of angles** are adjacent angles that are supplementary. The two sides that they do not share in common are opposite rays. The term *linear* tells us that a *line* is used to form this pair. Since the angles are supplementary, if $m\angle DBC = x$, then $m\angle CBA = (180 - x)$.

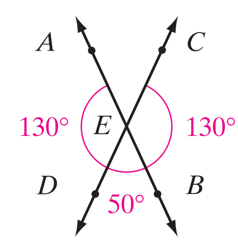
Vertical Angles



If two straight lines such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, then $\angle x$ and $\angle y$ share a common vertex at E but do not share a common side. Angles x and y are a pair of vertical angles. **Vertical angles** are two nonadjacent angles formed by two intersecting lines. In the diagram on the left, angles a and b are another pair of vertical angles.

If two lines intersect, four angles are formed that have no common interior point. In the diagram, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E. There are four linear pairs of angles:

$\angle AED$ and $\angle DEB$ $\quad$ $\angle BEC$ and $\angle CEA$
$\angle DEB$ and $\angle BEC$ $\quad$ $\angle CEA$ and $\angle AED$



The angles of each linear pair are supplementary.

- If $m\angle AED = 130$, then $m\angle DEB = 180 - 130 = 50$.
- If $m\angle DEB = 50$, then $m\angle BEC = 180 - 50 = 130$.

Therefore, $m\angle AED = m\angle BEC$.

DEFINITION

When two angles have equal measures, they are **congruent**.

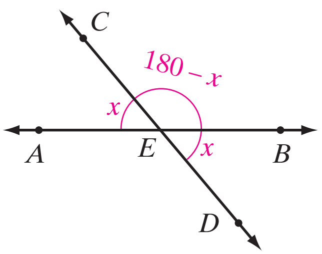
We use the symbol $\cong$ to represent the phrase "is congruent to." Since $m\angle AED = m\angle BEC$, we can write $\angle AED \cong \angle BEC$, read as "angle AED is congruent to angle BEC." There are different correct ways to indicate angles with equal measures:

1. The angle measures are equal: $m\angle BEC = m\angle AED$
2. The angles are congruent: $\angle BEC \cong \angle AED$

It would not be correct to say that the angles are equal, or that the angle measures are congruent.

Notice that we have just shown that the two vertical angles $\angle BEC$ and $\angle AED$ are congruent. If we were to draw and measure additional pairs of vertical angles, we would find in each case that the vertical angles would be equal in measure. No matter how many examples of a given situation we consider, however, we cannot *assume* that a conclusion that we draw from these examples will always be true. We must *prove* the conclusion. Statements that we prove are called **theorems**. We will use algebraic expressions and properties to write an informal proof of the following statement.

► **If two lines intersect, the vertical angles formed are equal in measure; that is, they are congruent.**



(1) If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, then $\angle AEB$ is a straight angle whose measure is 180°. Therefore, $m\angle AEC + m\angle CEB = 180$.

(2) Let $m\angle AEC = x$. Then $m\angle CEB = 180 - x$.

(3) Likewise, $\angle CED$ is a straight angle whose measure is 180°. Therefore, $m\angle CEB + m\angle BED = 180$.

(4) Since $m\angle CEB = 180 - x$, then $m\angle BED = 180 - (180 - x) = x$.

(5) Since both $m\angle AEC = x$ and $m\angle BED = x$, then $m\angle AEC = m\angle BED$; that is, $\angle AEC \cong \angle BED$.

EXAMPLE 1

The measure of the complement of an angle is 4 times the measure of the angle. Find the measure of the angle.

Solution Let x = measure of angle.

Then $4x$ = measure of complement of angle.

The sum of the measures of an angle and its complement is 90°.

$$x + 4x = 90$$
$$5x = 90$$
$$x = 18$$

Check The measure of the first angle is 18°.

The measure of the second angle is 4(18°) = 72°.

The sum of the measures of the angles is 18° + 72° = 90°.

Thus, the angles are complementary. ✔

Answer The measure of the angle is 18°.

Note: The unit of measure is very important in the solution of a problem. While it is not necessary to include the unit of measure in each step of the solution, each term in an equation must represent the same unit and the unit of measure must be included in the answer.

EXAMPLE 2

The measure of an angle is 40° more than the measure of its supplement. Find the measure of the angle.

Solution Let x = the measure of the supplement of the angle.

Then $x + 40$ = the measure of the angle.

The sum of the measures of an angle and its supplement is 180°.

$$x + (x + 40) = 180$$
$$2x + 40 = 180$$
$$2x = 140$$
$$x = 70$$
$$x + 40 = 110$$

Check The sum of the measures is $110° + 70° = 180°$ and 110° is 40° more than 70°. ✔

Answer The measure of the angle is 110°.

EXAMPLE 3

The algebraic expressions $5w - 20$ and $2w + 16$ represent the measures in degrees of a pair of vertical angles.

a. Find the value of w.

b. Find the measure of each angle.

Solution **a.** Vertical angles are equal in measure.

$$5w - 20 = 2w + 16$$
$$3w - 20 = 16$$
$$3w = 36$$
$$w = 12$$

b. $5w - 20 = 5(12) - 20 = 60 - 20 = 40$

$2w + 16 = 2(12) + 16 = 24 + 16 = 40$

Check Since each angle has a measure of 40°, the vertical angles are equal in measure. ✔

Answers **a.** $w = 12$ **b.** The measure of each angle is 40°.

EXERCISES

Writing About Mathematics

1. Show that supplementary angles are always two right angles or an acute angle and an obtuse angle.

2. The measures of three angles are 15°, 26°, and 49°. Are these angles complementary? Explain why or why not.

Developing Skills

In 3–6, answer each of the following questions for an angle with the given measure.

a. What is the measure of the complement of the angle?

b. What is the measure of the supplement of the angle?

c. The measure of the supplement of the angle is how much larger than the measure of its complement?

3. 15° **4.** 37° **5.** 67° **6.** $x°$

In 7–10, $\angle A$ and $\angle B$ are complementary. Find the measure of each angle if the measures of the two angles are represented by the given expressions. Solve the problem algebraically using an equation.

7. $m\angle A = x, m\angle B = 5x$

8. $m\angle A = x, m\angle B = x + 20$

9. $m\angle A = x, m\angle B = x - 40$

10. $m\angle A = y, m\angle B = 2y + 30$

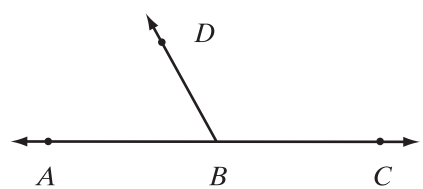
In 11–14, $\angle ABD$ and $\angle DBC$ are supplementary. Find the measure of each angle if the measures of the two angles are represented by the given expressions. Solve the problem algebraically using an equation.

11. $m\angle ABD = x, m\angle DBC = 3x$

12. $m\angle ABD = x, m\angle DBC = x + 80$

13. $m\angle DBC = x, m\angle ABD = x - 30$

14. $m\angle DBC = y, m\angle ABD = \frac{1}{4}y$



In 15–24, solve each problem algebraically using an equation.

15. Two angles are supplementary. The measure of one angle is twice as large as the measure of the other. Find the number of degrees in each angle.

16. The complement of an angle is 14 times as large as the angle. Find the measure of the complement.

17. The measure of the supplement of an angle is 40° more than the measure of the angle. Find the number of degrees in the supplement.

18. Two angles are complementary. One angle is twice as large as the other. Find the number of degrees in each angle.

19. The measure of the complement of an angle is one-ninth the measure of the angle. Find the measure of the angle.

20. Find the number of degrees in the measure of an angle that is 20° less than 4 times the measure of its supplement.

21. The difference between the measures of two supplementary angles is 80°. Find the measure of the larger of the two angles.

22. The complement of an angle measures 20° more than the angle. Find the number of degrees in the angle.

23. Find the number of degrees in an angle that measures 10° more than its supplement.

24. Find the number of degrees in an angle that measures 8° less than its complement.

25. The supplement of the complement of an acute angle is always:

(1) an acute angle (2) a right angle (3) an obtuse angle (4) a straight angle

In 26–28, $\overleftrightarrow{MN}$ and $\overleftrightarrow{RS}$ intersect at T.

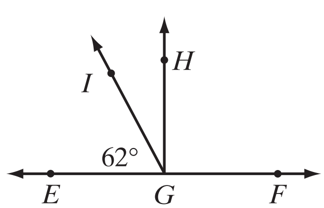
26. If $m\angle RTM = 5x$ and $m\angle NTS = 3x + 10$, find $m\angle RTM$.

27. If $m\angle MTS = 4x - 60$ and $m\angle NTR = 2x$, find $m\angle MTS$.

28. If $m\angle RTM = 7x + 16$ and $m\angle NTS = 3x + 48$, find $m\angle NTS$.

In 29–34, find the measure of each angle named, based on the given information.

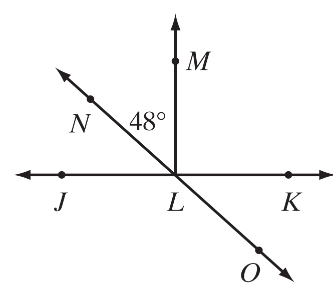
29. *Given*: $\overleftrightarrow{EF} \perp \overrightarrow{GH}$; $m\angle EGI = 62$.



a. Find $m\angle FGH$.

b. Find $m\angle HGI$.

30. *Given*: $\overleftrightarrow{JK} \perp \overrightarrow{LM}$; $\overleftrightarrow{NLO}$ is a line; $m\angle NLM = 48$.



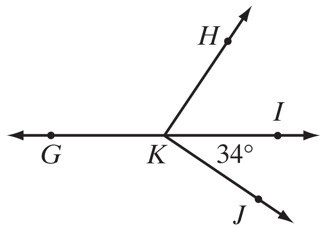
a. Find $m\angle JLN$.

b. Find $m\angle MLK$.

c. Find $m\angle KLO$.

d. Find $m\angle JLO$.

31. *Given*: $\angle GKH$ and $\angle HKI$ are a linear pair; $\overrightarrow{KH} \perp \overrightarrow{KJ}$; $m\angle IKJ = 34$.

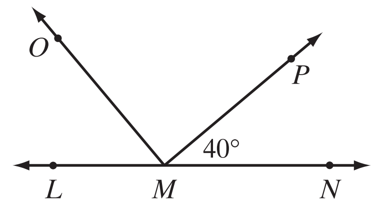


a. Find $m\angle HKI$.

b. Find $m\angle HKG$.

c. Find $m\angle GKJ$.

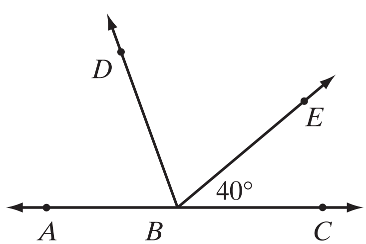
32. *Given*: $\overrightarrow{MO} \perp \overrightarrow{MP}$; $\overleftrightarrow{LMN}$ is a line; $m\angle PMN = 40$.



a. Find $m\angle PMO$.

b. Find $m\angle OML$.

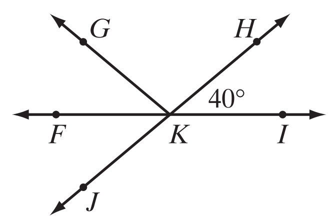
33. *Given*: $\overleftrightarrow{ABC}$ is a line; $m\angle EBC = 40$; $\angle ABD \cong \angle DBE$.



a. Find $m\angle ABD$.

b. Find $m\angle DBC$.

34. *Given*: $\overleftrightarrow{FI}$ intersects $\overleftrightarrow{JH}$ at K; $m\angle HKI = 40$; $m\angle FKG = m\angle FKJ$.



a. Find $m\angle FKJ$.

b. Find $m\angle FKG$.

c. Find $m\angle GKH$.

d. Find $m\angle JKI$.

In 35–38, sketch and label a diagram in each case and find the measure of each angle named.

35. $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at E; $m\angle AED = 20$. Find:

a. $m\angle CEB$ **b.** $m\angle BED$ **c.** $m\angle CEA$

36. $\angle PQR$ and $\angle RQS$ are complementary; $m\angle PQR = 30$; $\overleftrightarrow{RQT}$ is a line. Find:

a. $m\angle RQS$ **b.** $m\angle SQT$ **c.** $m\angle PQT$

37. $\overleftrightarrow{LM}$ intersects $\overleftrightarrow{PQ}$ at R. The measure of $\angle LRQ$ is 80 more than m$\angle LRP$. Find:

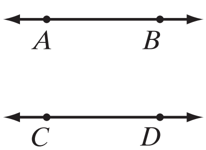
a. m$\angle LRP$ **b.** m$\angle LRQ$ **c.** m$\angle PRM$

38. $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ at E. Point F is in the interior of $\angle CEB$. The measure of $\angle CEF$ is 8 times the measure of $\angle FEB$. Find:

a. m$\angle FEB$ **b.** m$\angle CEF$ **c.** m$\angle AEF$

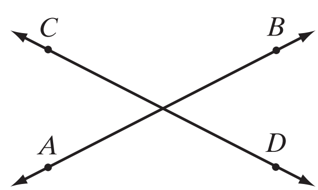
39. The angles, $\angle ABD$ and $\angle DBC$, form a linear pair and are congruent. What must be true about $\overleftrightarrow{ABC}$ and $\overleftrightarrow{BD}$?

7-3 ANGLES AND PARALLEL LINES

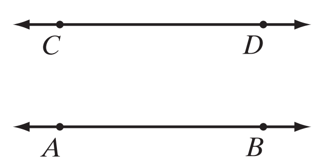


Not all lines in the same plane intersect. Two or more lines are called **parallel lines** if and only if the lines lie in the same plane but do not intersect.

In the figure on the left, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ lie in the same plane but do not intersect. Hence, we say that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. Using the symbol $\parallel$ for *is parallel to*, we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. When we speak of two parallel lines, we will mean two *distinct* lines. (In more advanced courses, you will see that a line is parallel to itself.) Line segments and rays are parallel if the lines that contain them are parallel. If two lines such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ lie in the same plane, they must be either intersecting lines or parallel lines, as shown in the following figures.

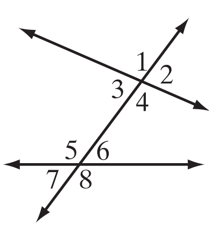


$\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$.



$\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$.

When two lines such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, they have no points in common. We can think of each line as a set of points. Hence, the intersection set of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is the empty set symbolized as $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \varnothing$.

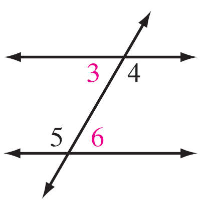


When two lines are cut by a third line, called a **transversal**, two sets of angles, each containing four angles, are formed. In the figure on the left:

- Angles 3, 4, 5, 6 are called **interior angles**.
- Angles 1, 2, 7, 8 are called **exterior angles**.
- Angles 4 and 5 are interior angles on opposite sides of the transversal and do not have the same vertex. They are called **alternate interior angles**. Angles 3 and 6 are another pair of alternate interior angles.

- Angles 1 and 8 are exterior angles on opposite sides of the transversal and do not have the same vertex. They are called **alternate exterior angles**. Angles 2 and 7 are another pair of alternate exterior angles.
- Angles 4 and 6 are **interior angles on the same side of the transversal**. Angles 3 and 5 are another pair of interior angles on the same side of the transversal.
- Angles 1 and 5 are on the same side of the transversal, one interior and one exterior, and at different vertices. They are called **corresponding angles**. Other pairs of corresponding angles are 2 and 6, 3 and 7, 4 and 8.

Alternate Interior Angles and Parallel Lines



In the figure, a transversal intersects two parallel lines, forming a pair of alternate interior angles, ∠3 and ∠6. If we measure ∠3 and ∠6 with a protractor, we will find that each angle measures 60°. Here, alternate interior angles 3 and 6 have equal measures, and ∠3 ≅ ∠6. If we draw other pairs of parallel lines intersected by transversals, we will find again that pairs of alternate interior angles have equal measures. Yet, we would be hard-pressed to prove that this is *always* true. Therefore, we accept, without proof, the following statement:

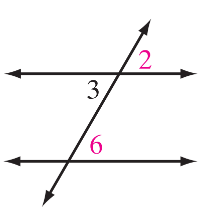
► **If two parallel lines are cut by a transversal, then the alternate interior angles that are formed have equal measures, that is, they are congruent.**

Note that ∠4 and ∠5 are another pair of alternate interior angles formed by a transversal that intersects the parallel lines. Therefore, ∠4 ≅ ∠5.

Corresponding Angles and Parallel Lines

If two lines are cut by a transversal, four pairs of corresponding angles are formed.

One such pair of corresponding angles is ∠2 and ∠6, as shown in the figure on the left. If the original two lines are parallel, do these corresponding angles have equal measures? We are ready to prove in an informal manner that they do.



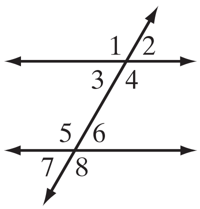
(1) Let m∠2 = x.

(2) If m∠2 = x, then m∠3 = x (because ∠2 and ∠3 are vertical angles, and vertical angles are congruent).

(3) If m∠3 = x, then m∠6 = x (because ∠3 and ∠6 are alternate interior angles of parallel lines, and alternate interior angles of parallel lines are congruent).

(4) Therefore m∠2 = m∠6 (because the measure of each angle is x).

The four steps on page 259 serve as a proof of the following theorem:

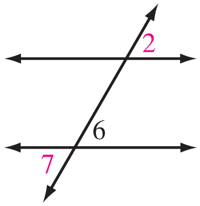


> **Theorem.** If two parallel lines are cut by a transversal, then the corresponding angles formed have equal measures, that is, they are congruent.

Note that this theorem is true for each pair of corresponding angles:

$$\angle 1 \cong \angle 5; \angle 2 \cong \angle 6; \angle 3 \cong \angle 7; \angle 4 \cong \angle 8$$

Alternate Exterior Angles and Parallel Lines



If two parallel lines are cut by a transversal, we can prove informally that the alternate exterior angles formed have equal measures. One such pair of alternate exterior angles consists of $\angle 2$ and $\angle 7$, as shown in the figure on the left.

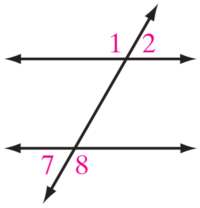
(1) Let $m\angle 2 = x$.

(2) If $m\angle 2 = x$, then $m\angle 6 = x$ (because $\angle 2$ and $\angle 6$ are corresponding angles of parallel lines, proved to have the same measure).

(3) If $m\angle 6 = x$, then $m\angle 7 = x$ (because $\angle 6$ and $\angle 7$ are vertical angles, previously proved to have the same measure).

(4) Therefore $m\angle 2 = m\angle 7$ (because the measure of each angle is x).

These four steps serve as a proof of the following theorem:

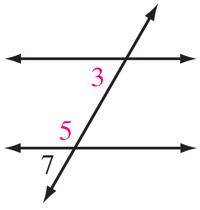


> **Theorem.** If two parallel lines are cut by a transversal, then the alternate exterior angles formed have equal measures, that is, they are congruent.

Note that this theorem is true for each pair of alternate exterior angles:

$$\angle 1 \cong \angle 8; \angle 2 \cong \angle 7.$$

Interior Angles on the Same Side of the Transversal



When two parallel lines are cut by a transversal, we can prove informally that the sum of the measures of the interior angles on the same side of the transversal is 180°. One such pair of interior angles on the same side of the transversal consists of $\angle 3$ and $\angle 5$, as shown in the figure on the left.

(1) $m\angle 5 + m\angle 7 = 180$ ($\angle 5$ and $\angle 7$ are supplementary angles).

(2) $m\angle 7 = m\angle 3$ ($\angle 7$ and $\angle 3$ are corresponding angles).

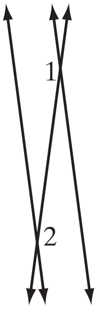
(3) $m\angle 5 + m\angle 3 = 180$ (by substituting $m\angle 3$ for its equal, $m\angle 7$).

These three steps serve as a proof of the following theorem:

> **Theorem.** If two parallel lines are cut by a transversal, then the sum of the measures of the interior angles on the same side of the transversal is 180°.

EXAMPLE 1

In the figure, the parallel lines are cut by a transversal. If $m\angle 1 = (5x - 10)$ and $m\angle 2 = (3x + 60)$, find the measures of $\angle 1$ and $\angle 2$.



Solution Since the lines are parallel, the alternate interior angles, $\angle 1$ and $\angle 2$, have equal measures. Write and solve an equation using the algebraic expressions for the measures of these angles:

$$\begin{aligned} 5x - 10 &= 3x + 60 \\ 5x &= 3x + 70 \\ 2x &= 70 \\ x &= 35 \end{aligned}$$

$$5x - 10 = 5(35) - 10 = 175 - 10 = 165$$
$$3x + 60 = 3(35) + 60 = 105 + 60 = 165$$

Answer $m\angle 1 = 165$ and $m\angle 2 = 165$.

EXERCISES

Writing About Mathematics

1. Give an example of a pair of lines that are neither parallel nor intersecting.

2. If a transversal is perpendicular to one of two parallel lines, is it perpendicular to the other? Prove your answer using definitions and theorems given in this chapter.

Developing Skills

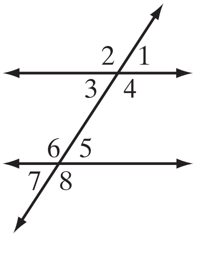
In 3–10, the figure below shows two parallel lines cut by a transversal. For each given measure, find the measures of the other seven angles.

3. $m\angle 3 = 80$ **4.** $m\angle 6 = 150$

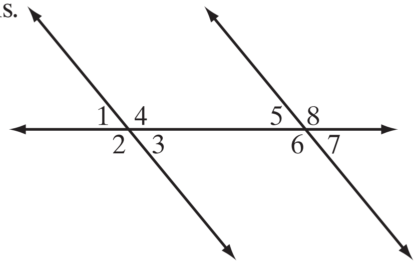
5. $m\angle 5 = 60$ **6.** $m\angle 1 = 75$

7. $m\angle 2 = 55$ **8.** $m\angle 4 = 10$

9. $m\angle 7 = 2$ **10.** $m\angle 8 = 179$



In 11–15, the figure below shows two parallel lines cut by a transversal. In each exercise, find the measures of all eight angles under the given conditions.



11. $m\angle 3 = 2x + 40$ and $m\angle 7 = 3x + 27$

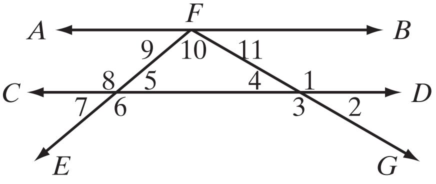
12. $m\angle 4 = 4x - 10$ and $m\angle 6 = x + 80$

13. $m\angle 4 = 3x + 40$ and $m\angle 5 = 2x$

14. $m\angle 4 = 2x - 10$ and $m\angle 2 = x + 60$

15. $\angle 8 \cong \angle 1$

16. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle 5 = 40$, and $m\angle 4 = 30$, find the measures of the other angles in the figure.



In 17–22, tell whether each statement is always, sometimes, or never true.

17. If two distinct lines intersect, then they are parallel.

18. If two distinct lines do not intersect, then they are parallel.

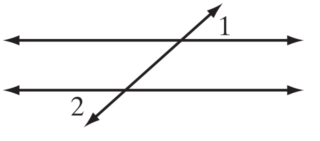
19. If two angles are alternate interior angles, then they are on opposite sides of the transversal.

20. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

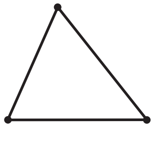
21. If two parallel lines are cut by a transversal, then the alternate interior angles are complementary.

22. If two parallel lines are cut by a transversal, then the corresponding angles are supplementary.

23. In the figure on the right, two parallel lines are cut by a transversal. Write an informal proof that $\angle 1$ and $\angle 2$ have equal measures.



7-4 TRIANGLES



A **polygon** is a plane figure that consists of line segments joining three or more points. Each line segment is a **side** of the polygon and the endpoints of each side are called **vertices**. Each vertex is the endpoint of exactly two sides and no two sides have any other points in common.

When three points that are not all on the same line are joined in pairs by line segments, the figure formed is a **triangle**. Each of the given points is the vertex of an angle of the triangle, and each line segment is a side of the triangle. There are many practical uses of the triangle, especially in construction work such as the building of bridges, radio towers, and airplane wings, because the triangle is a rigid figure. The shape of a triangle cannot be changed without changing the length of at least one of its sides.

We begin our discussion of the triangle by considering triangle ABC, shown below, which is symbolized as $\triangle ABC$. In $\triangle ABC$, points A, B, and C are the vertices, and $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are the sides. Angles A, B, and C of the triangle are symbolized as $\angle A$, $\angle B$, and $\angle C$.

We make the following observations:

1. Side $\overline{AB}$ is included between $\angle A$ and $\angle B$.

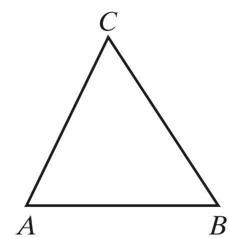
2. Side $\overline{BC}$ is included between $\angle B$ and $\angle C$.

3. Side $\overline{CA}$ is included between $\angle C$ and $\angle A$.

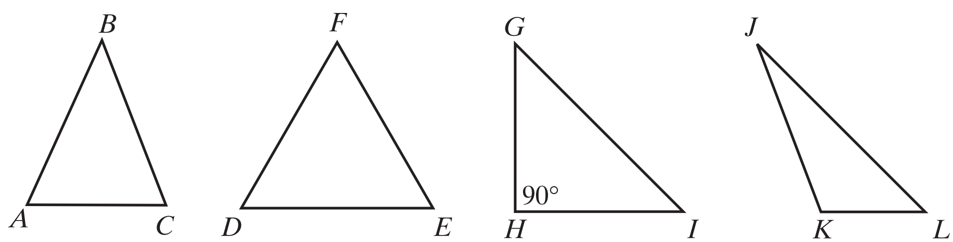
4. Angle A is included between sides $\overline{AB}$ and $\overline{CA}$.

5. Angle B is included between sides $\overline{AB}$ and $\overline{BC}$.

6. Angle C is included between sides $\overline{CA}$ and $\overline{BC}$.



Classifying Triangles According to Angles



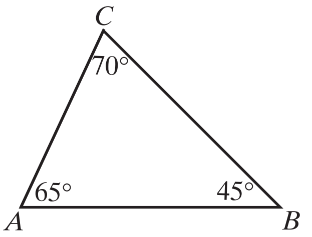
Acute triangle   Equiangular triangle   Right triangle   Obtuse triangle

- An **acute triangle** has three acute angles.
- An **equiangular triangle** has three angles equal in measure.
- A **right triangle** has one right angle.
- An **obtuse triangle** has one obtuse angle.

In right triangle GHI above, the two sides of the triangle that form the right angle, $\overline{GH}$ and $\overline{HI}$, are called the **legs** of the right triangle. The side opposite the right angle, $\overline{GI}$, is called the **hypotenuse**.

Sum of the Measures of the Angles of a Triangle

When we change the shape of a triangle, changes take place also in the measures of its angles. Is there any relationship among the measures of the angles of a triangle that does not change? Let us see.



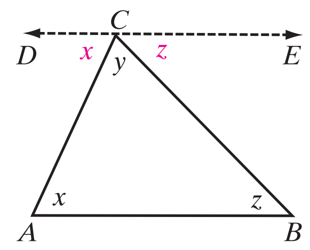
Draw several triangles of different shapes. Measure the three angles of each triangle and find the sum of these measures. For example, in $\triangle ABC$ on the right,

$$m\angle A + m\angle B + m\angle C = 65 + 45 + 70 = 180.$$

If you measured accurately, you should have found that in each triangle that you drew and measured, the sum of the measures of the three angles is 180°, regardless of its size or shape.

We can write an informal algebraic proof of the following statement:

► **The sum of the measures of the angles of a triangle is 180°.**



(1) In $\triangle ABC$, let $m\angle A = x$, $m\angle ACB = y$, and $m\angle B = z$.

(2) Let $\overleftrightarrow{DCE}$ be a line parallel to $\overleftrightarrow{AB}$.

(3) Since $\angle DCE$ is a straight angle, $m\angle DCE = 180$.

(4) $m\angle DCE = m\angle DCA + m\angle ACB + m\angle BCE = 180.$

(5) $m\angle DCA = m\angle A = x$.

(6) $m\angle ACB = y$.

(7) $m\angle BCE = m\angle B = z$.

(8) Substitute from statements (5), (6), and (7) in statement (4): $x + y + z = 180$.

EXAMPLE 1

Find the measure of the third angle of a triangle if the measures of two of the angles are 72.6° and 84.2°.

Solution Subtract the sum of the known measures from 180:

$$180 - (72.6 + 84.2) = 23.2$$

Answer 23.2°

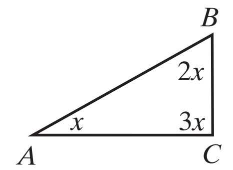
EXAMPLE 2

In $\triangle ABC$, the measure of $\angle B$ is twice the measure of $\angle A$, and the measure of $\angle C$ is 3 times the measure of $\angle A$. Find the number of degrees in each angle of the triangle.

Solution Let x = the number of degrees in $\angle A$.

Then $2x$ = the number of degrees in $\angle B$.

Then $3x$ = the number of degrees in $\angle C$.



The sum of the measures of the angles of a triangle is 180°.

$$x + 2x + 3x = 180$$
$$6x = 180$$
$$x = 30$$
$$2x = 60$$
$$3x = 90$$

Check

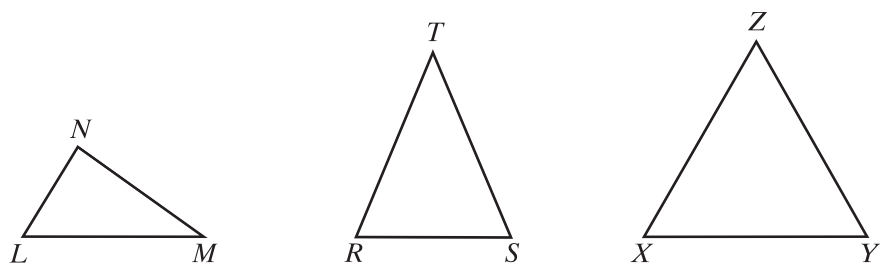
$60 = 2(30)$

$90 = 3(30)$

$30 + 60 + 90 = 180$ ✔

Answer $m\angle A = 30, m\angle B = 60, m\angle C = 90$

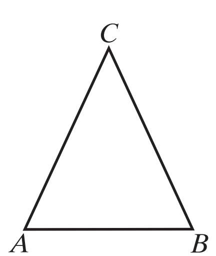
Classifying Triangles According to Sides



Scalene triangle　Isosceles triangle　Equilateral triangle

- A **scalene triangle** has no sides equal in length.
- An **isosceles triangle** has two sides equal in length.
- An **equilateral triangle** has three sides equal in length.

Isosceles Triangles



In isosceles triangle ABC, shown on the left, the two sides that are equal in measure, $\overline{AC}$ and $\overline{BC}$, are called the **legs**. The third side, $\overline{AB}$, is the **base**.

Two line segments that are equal in measure are said to be **congruent**. The angle formed by the two congruent sides, $\angle C$, is called the **vertex angle**. The two angles at the endpoints of the base, $\angle A$ and $\angle B$, are the **base angles**. In isosceles triangle ABC, if we measure the base angles, $\angle A$ and $\angle B$, we find that each angle contains 65°. Therefore, $m\angle A = m\angle B$. Similarly, if we measure the base angles in any other isosceles triangle, we find that they are equal in measure. Thus, we will accept the truth of the following statement:

► **The base angles of an isosceles triangle are equal in measure; that is, they are congruent.**

This statement may be rephrased in a variety of ways. For example:

1. If a triangle is isosceles, then its base angles are congruent.
2. If two sides of a triangle are equal in measure, then the angles opposite these sides are equal in measure.

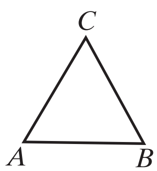
The following statement is also true:

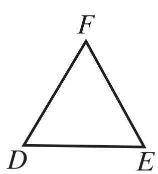
► **If two angles of a triangle are equal in measure, then the sides opposite these angles are equal in measure and the triangle is an isosceles triangle.**

This statement may be rephrased as follows:

1. If two angles of a triangle are congruent, then the sides opposite these angles are congruent.
2. If two angles of a triangle have equal measures, then the sides opposite these angles have equal measures.

Equilateral Triangles





Triangle ABC is an equilateral triangle. Since $AB = BC$, $m\angle C = m\angle A$; also, since $AC = BC$, $m\angle B = m\angle A$. Therefore, $m\angle A = m\angle B = m\angle C$. In an equilateral triangle, the measures of all of the angles are equal.

In $\triangle DEF$, all of the angles are equal in measure. Since $m\angle D = m\angle E$, $EF = DF$; also, since $m\angle D = m\angle F$, $EF = DE$. Therefore, $DE = EF = DF$, and $\triangle DEF$ is equilateral.

► **If a triangle is equilateral, then it is equiangular.**

Properties of Special Triangles

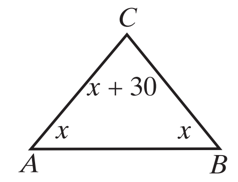
1. If two sides of a triangle are equal in measure, the angles opposite these sides are also equal in measure. (The base angles of an isosceles triangle are equal in measure.)
2. If two angles of a triangle are equal in measure, the sides opposite these angles are also equal in measure.
3. All of the angles of an equilateral triangle are equal in measure. (An equilateral triangle is equiangular.)
4. If three angles of a triangle are equal in measure, the triangle is equilateral. (An equiangular triangle is equilateral.)

EXAMPLE 3

In isosceles triangle ABC, the measure of vertex angle C is 30° more than the measure of each base angle. Find the number of degrees in each angle of the triangle.

Solution Let x = number of degrees in one base angle, A.

Then x = number of degrees in the other base angle, B, and $x + 30$ = number of degrees in the vertex angle, C.



The sum of the measures of the angles of a triangle is 180°.

$$x + x + x + 30 = 180$$
$$3x + 30 = 180$$
$$3x = 150$$
$$x = 50$$
$$x + 30 = 80$$

Check $50 + 50 + 80 = 180$ ✔

Answer $m\angle A = 50, m\angle B = 50, m\angle C = 80$

EXERCISES

Writing About Mathematics

1. Ayyam said that if the sum of the measures of two angles of a triangle is equal to the measure of the third angle, the triangle is a right triangle. Prove or disprove Ayyam's statement.

2. Janice said that if two angles of a triangle each measure 60°, then the triangle is equilateral. Prove or disprove Janice's statement.

Developing Skills

In 3–5, state, in each case, whether the angles with the given measures can be the three angles of the same triangle.

3. 30°, 70°, 80° **4.** 70°, 80°, 90° **5.** 30°, 110°, 40°

In 6–9, find, in each case, the measure of the third angle of the triangle if the measures of two angles are:

6. 60°, 40° **7.** 100°, 20° **8.** 54.5°, 82.3° **9.** $24\frac{1}{4}°, 81\frac{1}{2}°$

10. What is the measure of each angle of an equiangular triangle?

11. Can a triangle have: **a.** two right angles? **b.** two obtuse angles? **c.** one right and one obtuse angle? Explain why or why not.

12. What is the sum of the measures of the two acute angles of a right triangle?

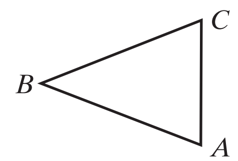
13. In $\triangle ABC$, $AC = 4$, $CB = 6$, and $AB = 6$.

a. What type of triangle is $\triangle ABC$?

b. Name two angles in $\triangle ABC$ whose measures are equal.

c. Why are these angles equal in measure?

d. Name the legs, base, base angles and vertex angle of this triangle.



14. In $\triangle RST$, $m\angle R = 70$ and $m\angle T = 40$.

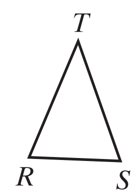
a. Find the measure of $\angle S$.

b. Name two sides in $\triangle RST$ that are congruent.

c. Why are the two sides congruent?

d. What type of triangle is $\triangle RST$?

e. Name the legs, base, base angles, and vertex angle of this triangle.



15. Find the measure of the vertex angle of an isosceles triangle if the measure of each base angle is:

a. 80° **b.** 55° **c.** 42° **d.** $22\frac{1}{2}$° **e.** 51.5°

16. Find the measure of each base angle of an isosceles triangle if the measure of the vertex angle is:

a. 20° **b.** 50° **c.** 76° **d.** 100° **e.** 65°

17. What is the number of degrees in each acute angle of an isosceles right triangle?

18. If a triangle is equilateral, what is the measure of each angle?

Applying Skills

19. The measure of each base angle of an isosceles triangle is 7 times the measure of the vertex angle. Find the measure of each angle of the triangle.

20. The measure of each of the congruent angles of an isosceles triangle is one-half of the measure of the vertex angle. Find the measure of each angle of the triangle.

21. The measure of the vertex angle of an isosceles triangle is 3 times the measure of each base angle. Find the number of degrees in each angle of the triangle.

22. The measure of the vertex angle of an isosceles triangle is 15° more than the measure of each base angle. Find the number of degrees in each angle of the triangle.

23. The measure of each of the congruent angles of an isosceles triangle is 6° less than the measure of the vertex angle. Find the measure of each angle of the triangle.

24. The measure of each of the congruent angles of an isosceles triangle is 9° less than 4 times the vertex angle. Find the measure of each angle of the triangle.

25. In $\triangle ABC$, $m\angle A = x$, $m\angle B = x + 45$, and $m\angle C = 3x - 15$.

a. Find the measures of the three angles.

b. What kind of triangle is $\triangle ABC$?

26. The measures of the three angles of $\triangle DEF$ can be represented by $(x + 30)°$, $2x°$, and $(4x - 60)°$.

a. What is the measure of each angle?

b. What kind of triangle is $\triangle DEF$?

27. In a triangle, the measure of the second angle is 3 times the measure of the first angle, and the measure of the third angle is 5 times the measure of the first angle. Find the number of degrees in each angle of the triangle.

28. In a triangle, the measure of the second angle is 4 times the measure of the first angle. The measure of the third angle is equal to the sum of the measures of the first two angles. Find the number of degrees in each angle of the triangle. What kind of triangle is it?

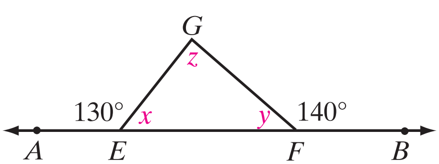
29. In a triangle, the measure of the second angle is 30° more than the measure of the first angle, and the measure of the third angle is 45° more than the measure of the first angle. Find the number of degrees in each angle of the triangle.

30. In a triangle, the measure of the second angle is 5° more than twice the measure of the first angle. The measure of the third angle is 35° less than 3 times the measure of the first angle. Find the number of degrees in each angle of the triangle.

31. $\overleftrightarrow{AEFB}$ is a straight line, $m\angle AEG = 130$, and $m\angle BFG = 140$.

a. Find $m\angle x$, $m\angle y$, and $m\angle z$.

b. What kind of a triangle is $\triangle EFG$?



32. In $\triangle RST$, $m\angle R = x$, $m\angle S = x + 30$, $m\angle T = x - 30$.

a. Find the measures of the three angles of the triangle.

b. What kind of a triangle is $\triangle RST$?

33. In $\triangle KLM$, $m\angle K = 2x$, $m\angle L = x + 30$, $m\angle M = 3x - 30$.

a. Find the measures of the three angles of the triangle.

b. What kind of a triangle is $\triangle KLM$?

Hands-On Activity 1: Constructing a Line Segment Congruent to a Given Segment

To **construct** a geometric figure means that a specific design is accurately made by using only two instruments: a **compass** used to draw a complete circle or part of a circle and a **straightedge** used to draw a straight line.

In this activity, you will learn how to construct a line segment congruent to a given segment, that is, construct a *copy* of a line segment.

STEP 1. Use the straightedge to draw a line segment. Label the endpoints A and B.

STEP 2. Use the straightedge to draw a ray and label the endpoint C.

STEP 3. Place the compass so that the point of the compass is at A and the point of the pencil is at B.

STEP 4. Keeping the opening of the compass unchanged, place the point at C and draw an arc that intersects the ray. Label this intersection D.

Result: $\overline{AB} \cong \overline{CD}$

a. Now that you know how to construct congruent line segments, explain how you can construct a line segment that is three times the length of a given segment.

b. Explain how to construct a line segment with length equal to the *difference* of two given segments.

c. Explain how to construct a line segment whose length is the sum of the lengths of two given line segments.

Hands-On Activity 2: Constructing an Angle Congruent to a Given Angle

In this activity, you will learn how to construct an angle congruent to a given angle, that is, construct a copy of an angle.

STEP 1. Use the straightedge to draw an acute angle. Label the vertex S.

STEP 2. Use the straightedge to draw a ray and label the endpoint M.

STEP 3. With the point of the compass at S, draw an arc that intersects each ray of $\angle S$. Label the point of intersection on one ray R and the point of intersection on the other ray T.

STEP 4. Using the same opening of the compass as was used in step 3, place the point of the compass at M and draw an arc that intersects the ray and extends beyond the ray. (Draw at least half of a circle.) Label the point of intersection L.

STEP 5. Place the point of the compass at R and the point of the pencil at T.

STEP 6. Without changing the opening of the compass, place the point at L and draw an arc that intersects the arc drawn in step 4. Label this point of intersection N.

STEP 7. Draw $\overrightarrow{MN}$.

Result: $\angle RST \cong \angle LMN$

a. Now that you know how to construct congruent angles, explain how you can construct an angle that is three times the measure of a given angle.

b. Explain how to construct an angle with a measure equal to the *difference* of two given angles.

c. Explain how to construct a triangle congruent to a given triangle using two sides and the included angle.

Hands-On Activity 3: Constructing a Perpendicular Bisector

In this activity, you will learn how to construct a **perpendicular bisector** of a line segment. A perpendicular bisector of a line segment is the line that divides a segment into two equal parts and is perpendicular to the segment.

STEP 1. Use the straightedge to draw a line segment. Label one endpoint A and the other C.

STEP 2. Open the compass so that the distance between the point and the pencil point is more than half of the length of $\overline{AC}$.

STEP 3. With the point of the compass at A, draw an arc above $\overline{AC}$ and an arc below $\overline{AC}$.

STEP 4. With the same opening of the compass and the point of the compass at C, draw an arc above $\overline{AC}$ and an arc below $\overline{AC}$ that intersect the arcs drawn in step 3. Call one of these intersections E and the other F.

STEP 5. Use the straightedge to draw $\overleftrightarrow{EF}$, intersecting $\overline{AC}$ at B.

Result: $\overleftrightarrow{EF}$ is perpendicular to $\overline{AC}$, and B is the midpoint of $\overline{AC}$.

a. What is true about $\angle EAB$ and $\angle FAB$? What is true about $\angle EAB$ and $\angle ECB$?

b. Explain how to construct an isosceles triangle with a vertex angle that is twice the measure of a given angle.

c. Explain how to construct an isosceles right triangle.

Hands-On Activity 4: Constructing an Angle Bisector

In this activity, you will learn how to construct an **angle bisector**. An angle bisector is the line that divides an angle into two congruent angles.

STEP 1. Use the straightedge to draw an acute angle. Label the vertex S.

STEP 2. With any convenient opening of the compass, place the point at S and draw an arc that intersects both rays of $\angle S$. Call one of the intersections R and the other T.

STEP 3. Place the point of the compass at R and draw an arc in the interior of $\angle S$.

STEP 4. With the same opening of the compass, place the point of the compass at T and draw an arc that intersects the arc drawn in step 3. Label the intersection of the arcs P.

STEP 5. Draw $\overrightarrow{SP}$.

Result: $\angle RSP \cong \angle PST$; $\overrightarrow{SP}$ bisects angle S.

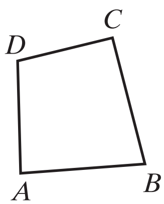
a. Now that you know how to construct an angle bisector, explain how you can construct an angle that is one and a half times the measure of a given angle.

b. Is it possible to use an angle bisector to construct a 90° angle? Explain.

c. Explain how to construct an isosceles triangle with a vertex angle that is congruent to a given angle.

7-5 QUADRILATERALS

In your study of mathematics you have learned many facts about polygons that have more than three sides. In this text you have frequently solved problems using the formulas for the perimeter and area of a rectangle. In this section we will review what you already know and use that knowledge to demonstrate the truth of many of the properties of polygons.

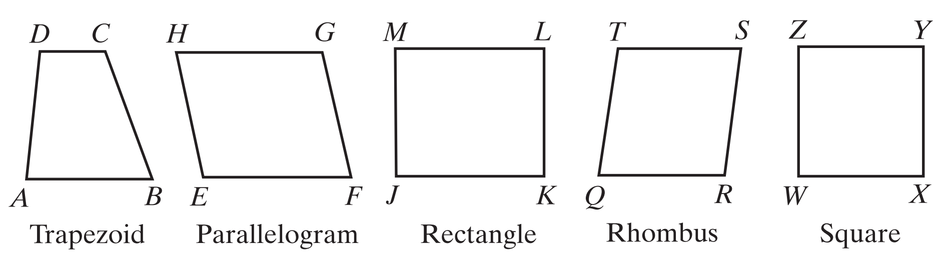


A **quadrilateral** is a polygon that has four sides. A point at which any two sides of the quadrilateral meet is a *vertex* of the quadrilateral. At each vertex, the two sides that meet form an angle of the quadrilateral. Thus, $ABCD$ on the left is a quadrilateral whose sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Its vertices are A, B, C, and D. Its angles are $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$.

In a quadrilateral, two angles whose vertices are the endpoints of a side are called **consecutive angles**. For example, in quadrilateral $ABCD$, $\angle A$ and $\angle B$ are consecutive angles because their vertices are the endpoints of a side, $\overline{AB}$. Other pairs of consecutive angles are $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$. Two angles that are not consecutive angles are called **opposite angles**; $\angle A$ and $\angle C$ are opposite angles, and $\angle B$ and $\angle D$ are opposite angles.

Special Quadrilaterals

When we vary the shape of the quadrilateral by making some of its sides parallel, by making some of its sides equal in length, or by making its angles right angles, we get different members of the family of quadrilaterals, as shown and named below:



A **trapezoid** is a quadrilateral in which two and only two opposite sides are parallel. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$. Parallel sides $\overline{AB}$ and $\overline{CD}$ are called the **bases** of the trapezoid.

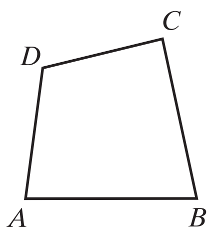
A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel. In parallelogram $EFGH$, $\overline{EF} \parallel \overline{GH}$ and $\overline{EH} \parallel \overline{FG}$. The symbol for a parallelogram is ▱.

A **rectangle** is a parallelogram in which all four angles are right angles. Rectangle $JKLM$ is a parallelogram in which $\angle J$, $\angle K$, $\angle L$, and $\angle M$ are right angles.

A **rhombus** is a parallelogram in which all sides are of equal length. Rhombus $QRST$ is a parallelogram in which $QR = RS = ST = TQ$.

A **square** is a rectangle in which all sides are of equal length. It is also correct to say that a square is a rhombus in which all angles are right angles. Therefore, square $WXYZ$ is also a parallelogram in which $\angle W$, $\angle X$, $\angle Y$, and $\angle Z$ are right angles, and $WX = XY = YZ = ZW$.

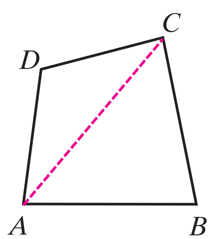
The Angles of a Quadrilateral



Draw a large quadrilateral like the one shown at the left and measure each of its four angles. Is the sum 360°? It should be. If we do the same with several other quadrilaterals of different shapes and sizes, is the sum of the four measures 360° in each case? It should be. Relying on what you have just verified by experimentation, it seems reasonable to make the following statement:

► **The sum of the measures of the angles of a quadrilateral is 360°.**

To prove informally that this statement is true, we draw diagonal $\overline{AC}$, whose endpoints are the vertices of the two opposite angles, $\angle A$ and $\angle C$. Then:



(1) Diagonal $\overline{AC}$ divides quadrilateral $ABCD$ into two triangles, $\triangle ABC$ and $\triangle ADC$.

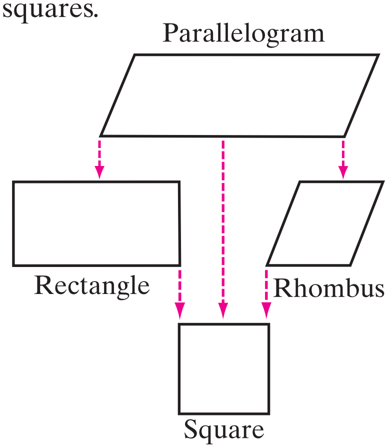
(2) The sum of the measures of the angles of $\triangle ABC$ is 180°, and the sum of the measures of the angles of $\triangle ADC$ is 180°.

(3) The sum of the measures of all the angles of $\triangle ABC$ and $\triangle ADC$ together is 360°.

(4) Therefore, $m\angle A + m\angle B + m\angle C + m\angle D = 360$.

The Family of Parallelograms

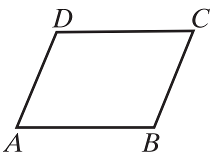
Listed below are some relationships that are true for the family of parallelograms that includes rectangles, rhombuses, and squares.



1. All rectangles, rhombuses, and squares are parallelograms. Therefore any property of a parallelogram must also be a property of a rectangle, a rhombus, or a square.

2. A square is also a rectangle. Therefore, any property of a rectangle must also be a property of a square.

3. A square is also a rhombus. Therefore, any property of a rhombus must also be a property of a square.



In parallelogram $ABCD$ at the left, opposite sides are parallel. Thus, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. The following statements are true for any parallelograms.

► **Opposite sides of a parallelogram are congruent.**

Here, $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$. Therefore $AB = DC$ and $AD = BC$.

► **Opposite angles of a parallelogram are congruent.**

Here, $\angle A \cong \angle C$ and $\angle B \cong \angle D$. Therefore $m\angle A = m\angle C$ and $m\angle B = m\angle D$.

► **Consecutive angles of a parallelogram are supplementary.**

Here, $m\angle A + m\angle B = 180$, $m\angle B + m\angle C = 180$, and so forth.

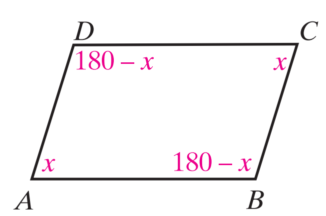
Since rhombuses, rectangles, and squares are also parallelograms, these statements will be true for any rhombus, any rectangle, and any square.

Informal Proofs for Statements About Angles in a Parallelogram

The statements about angles formed by parallel lines cut by a transversal that were shown to be true in Section 3 of this chapter can now be used to establish the relationships among the measures of the angles of a parallelogram.

(1) In parallelogram $ABCD$, $\overline{AB}$ and $\overline{DC}$ are segments of parallel lines cut by transversal $\overline{AD}$.

(2) Let $m\angle A = x$. Then $m\angle D = 180 - x$ because $\angle A$ and $\angle D$ are interior angles on the same side of a transversal, and these angles have been shown to be supplementary.



(3) In parallelogram $ABCD$, $\overline{AD}$ and $\overline{BC}$ are segments of parallel lines cut by transversal $\overline{AB}$.

(4) Since $m\angle A = x$, $m\angle B = 180 - x$ because $\angle A$ and $\angle B$ are interior angles on the same side of a transversal, and these angles have been shown to be supplementary.

(5) Similarly, $\overline{AD}$ and $\overline{BC}$ are segments of parallel lines cut by transversal $\overline{DC}$. Since $m\angle D = 180 - x$, $m\angle C = 180 - (180 - x) = x$ because $\angle D$ and $\angle C$ are interior angles on the same side of the transversal.

Therefore, the consecutive angles of a parallelogram are supplementary:

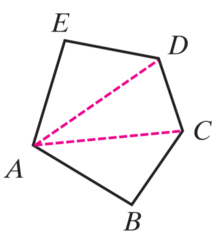
$$m\angle A + m\angle D = x + (180 - x) = 180$$
$$m\angle A + m\angle B = x + (180 - x) = 180$$
$$m\angle C + m\angle B = x + (180 - x) = 180$$
$$m\angle C + m\angle D = x + (180 - x) = 180$$

Also, the opposite angles of a parallelogram have equal measures:

$$m\angle A = x \text{ and } m\angle C = x \quad \text{or} \quad m\angle A = m\angle C$$

$$m\angle B = 180 - x \text{ and } m\angle D = 180 - x \quad \text{or} \quad m\angle B = m\angle D$$

Polygons and Angles

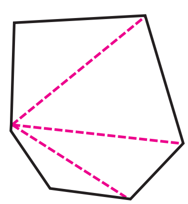


We can use the sum of the measures of the interior angles of a triangle to find the sum of the measures of the interior angles of any polygon.

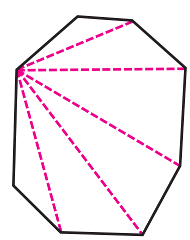
Polygon $ABCDE$ is a pentagon, a polygon with five sides. From vertex A, we draw diagonals to vertices C and D, the vertices that are not adjacent to A. These diagonals divide the pentagon into three triangles. The sum of the measures of the interior angles of $ABCDE$ is the sum of the measures of $\triangle ABC$, $\triangle ACD$, and $\triangle ADE$.

The sum of the measures of the interior angles of $ABCDE = 3(180°) = 540°$.

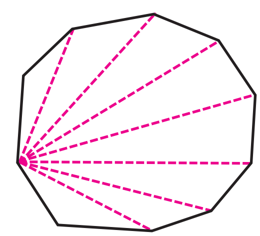
We can use this same method to find the sum of the interior angles of any polygon of more than three sides.



Hexagon
$4(180°) = 720°$



Octagon
$6(180°) = 1{,}080°$



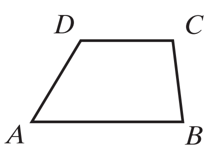
Decagon
$8(180°) = 1{,}440°$

In each case, the number of triangles into which the polygon can be divided is 2 fewer than the number of sides of the polygon. In general:

► **The sum of the measures of the interior angles of an *n*-sided polygon is $180(n - 2)$.**

Trapezoids

A trapezoid has one and only one pair of parallel lines. Each of the two parallel sides is called a *base* of the trapezoid. Therefore, we can use what we know about angles formed when parallel lines are cut by a transversal to demonstrate some facts about the angles of a trapezoid.

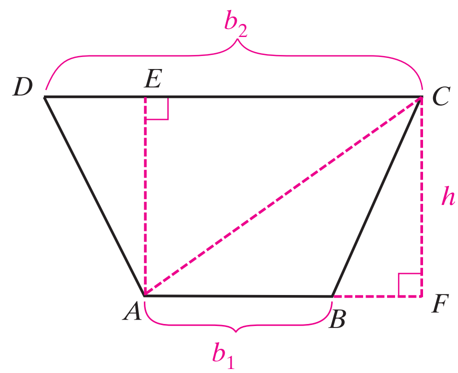


Quadrilateral $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$. Since $\angle DAB$ and $\angle CDA$ are interior angles on the same side of transversal $\overline{DA}$, they are supplementary. Also, $\angle CBA$ and $\angle DCB$ are interior angles on the same side of transversal $\overline{CB}$, and they are supplementary.

In a trapezoid, the parallel sides can never be congruent. But the nonparallel sides can be congruent. A trapezoid in which the nonparallel sides are congruent is called an **isosceles trapezoid**. In an isosceles trapezoid, the base angles, the two angles whose vertices are the endpoints of the same base, are congruent.

To write a formula for the area of trapezoid $ABCD$, draw diagonal $\overline{AC}$ and altitudes $\overline{AE}$ and $\overline{CF}$. Since $AFCE$ is a rectangle $AE = CF$.

Let $AB = b_1$, $CD = b_2$, and $AE = CF = h$.



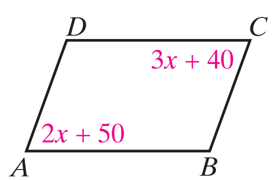
$$\text{Area of triangle } ABC = \tfrac{1}{2}b_1h$$
$$\text{Area of triangle } ACD = \tfrac{1}{2}b_2h$$
$$\text{Area of trapezoid } ABCD = \tfrac{1}{2}b_1h + \tfrac{1}{2}b_2h = \tfrac{1}{2}h(b_1 + b_2)$$

EXAMPLE 1

$ABCD$ is a parallelogram where $m\angle A = 2x + 50$ and $m\angle C = 3x + 40$.

a. Find the value of x.

b. Find the measure of each angle.



Solution **a.** In $\square ABCD$, $m\angle C = m\angle A$ because the opposite angles of a parallelogram have equal measure. Thus:

$$3x + 40 = 2x + 50$$
$$x = 10$$

b. By substitution:

$$m\angle A = 2x + 50 = 2(10) + 50 = 70$$
$$m\angle B = 180 - m\angle A = 180 - 70 = 110$$
$$m\angle C = 3x + 40 = 3(10) + 40 = 70$$
$$m\angle D = 180 - m\angle C = 180 - 70 = 110$$

Answer **a.** $x = 10$

b. $m\angle A = 70$

$m\angle B = 110$

$m\angle C = 70$

$m\angle D = 110$

EXERCISES

Writing About Mathematics

1. Adam said that if a quadrilateral has four equal angles then the parallelogram is a rectangle. Do you agree with Adam? Explain why or why not.

2. Emmanuel said that if a parallelogram has one right angle then the parallelogram is a rectangle. Do you agree with Emmanuel? Explain why or why not.

Developing Skills

In 3–8, in each case, is the statement true or false?

3. If a polygon is a trapezoid, it is a quadrilateral.

4. If a polygon is a rectangle, it is a parallelogram.

5. If a polygon is a rhombus, it is a parallelogram.

6. If a polygon is a rhombus, it is a square.

7. If a polygon is a square, then it is a rhombus.

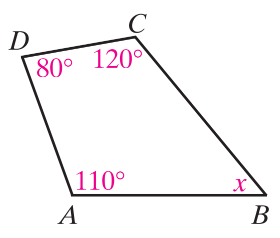
8. If two angles are opposite angles of a parallelogram, they are congruent.

In 9–12, the angle measures are represented in each quadrilateral. In each case:

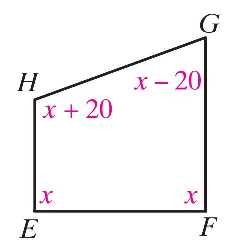
a. Find the value of x.

b. Find the measure of each angle of the quadrilateral.

9.



10.



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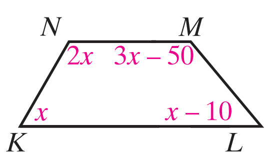

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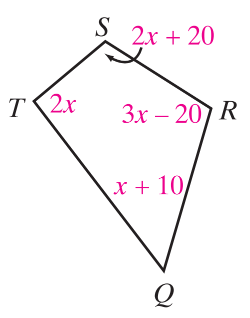
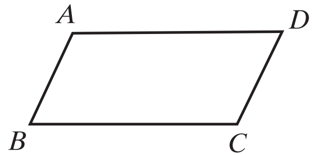


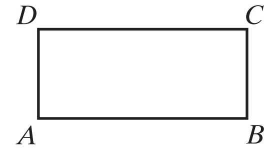
In 13 and 14, polygon *ABCD* is a parallelogram.



13. $AB = 3x + 8$ and $DC = x + 12$. Find *AB* and *DC*.

14. $m\angle A = 5x - 40$ and $m\angle C = 3x + 20$. Find $m\angle A$, $m\angle B$, $m\angle C$, and $m\angle D$.

In 15 and 16, polygon *ABCD* is a rectangle.



15. $BC = 4x - 5$, $AD = 2x + 3$. Find *BC* and *AD*.

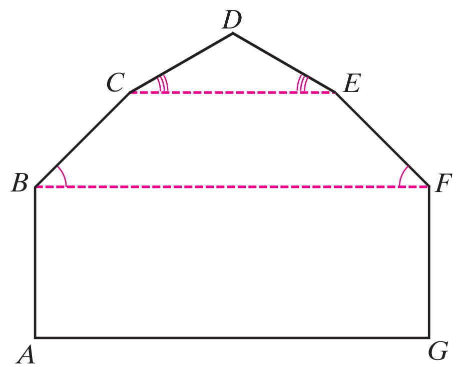
16. $m\angle A = 5x - 10$. Find the value of *x*.

17. *ABCD* is a square. If $AB = 8x - 6$ and $BC = 5x + 12$, find the length of each side of the square.

18. In rhombus *KLMN*, $KL = 3x$ and $LM = 2(x + 3)$. Find the length of each side of the rhombus.

Applying Skills

19. One side of a barn is in the shape of a seven-sided polygon called a heptagon. The heptagon, *ABCDEFG,* can be divided into rectangle *ABFG*, isosceles trapezoid *BCEF*, and isosceles triangle *CDE*, as shown in the diagram.



a. Find the sum of the measures of the interior angles of *ABCDEFG* by using the sum of the measures of the angles of the two quadrilaterals and the triangle.

b. Sketch the heptagon on your answer paper and show how it can be divided into triangles by drawing diagonals from *A*. Use these triangles to find the sum of the measures of the interior angles of *ABCDEFG*.

c. If the measure of each lower base angle of trapezoid *BCEF* is 45° and the measure of each base angle of isosceles triangle *CDE* is 30°, find the measure of each angle of heptagon *ABCDEFG*.

d. Find the sum of the measures of the interior angles of *ABCDEFG* by adding the angle measures found in **c.**

20. Cassandra had a piece of cardboard that was in the shape of an equilateral triangle. She cut an isosceles triangle from each vertex of the cardboard. The length of each leg of the triangles that she cut was one third of the length of a side of the original triangle.

a. Show that each of the triangles that Cassandra cut off is an equilateral triangle.

b. What is the measure of each angle of the remaining piece of cardboard?

c. What is the shape of the remaining piece of cardboard?

21. A piece of land is bounded by two parallel roads and two roads that are not parallel forming a trapezoid. Along the parallel roads the land measures 1.3 miles and 1.7 miles. The distance between the parallel roads, measured perpendicular to the roads, is 2.82 miles.

a. Find the area of the land. Express the area to the nearest tenth of a mile.

b. An acre is a unit of area often used to measure land. There are 640 acres in a square mile. Express, to the nearest hundred acres, the area of the land.

22. If possible, draw the following quadrilaterals. If it is not possible, state why.

a. A quadrilateral with four acute angles.

b. A quadrilateral with four obtuse angles.

c. A quadrilateral with one acute angle and three obtuse angles.

d. A quadrilateral with three acute angles and one obtuse angle.

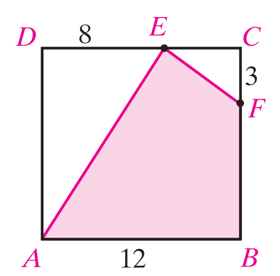
e. A quadrilateral with exactly three right angles.

7-6 AREAS OF IRREGULAR POLYGONS

Many polygons are irregular figures for which there is no formula for the area. However, the area of such figures can often be found by separating the figure into regions with known area formulas and adding or subtracting these areas to find the required area.

EXAMPLE 1

$ABCD$ is a square. E is a point on $\overline{DC}$ and F is a point on $\overline{BC}$. If $AB = 12$, $FC = 3$, and $DE = 8$, find the area of $ABFE$.



Solution Since $ABCD$ is a square and $AB = 12$, then $BC = 12$, $CD = 12$, and $DA = 12$.

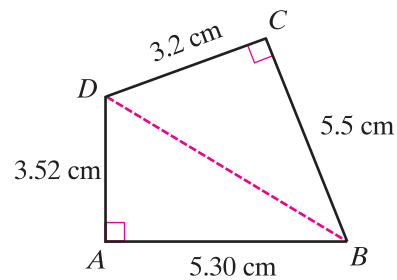
Therefore, $CE = CD - DE = 12 - 8 = 4$.

$$\begin{aligned} \text{Area of } ABFE &= \text{Area of } ABCD - \text{Area of } \triangle FCE - \text{Area of } \triangle EDA \\ &= 12(12) - \tfrac{1}{2}(3)(4) - \tfrac{1}{2}(8)(12) \\ &= 144 - 6 - 48 \\ &= 90 \end{aligned}$$

Answer 90 square units

EXAMPLE 2

ABCD is a quadrilateral. $AB = 5.30$ centimeters, $BC = 5.5$ centimeters, $CD = 3.2$ centimeters, and $DA = 3.52$ centimeters. Angle *A* and angle *C* are right angles. Find the area of *ABCD* to the correct number of significant digits.



Solution Draw $\overline{BD}$, separating the quadrilateral into two right triangles. In a right triangle, either leg is the base and the other leg is the altitude.

In $\triangle ABD$, $b = 5.30$, and $h = 3.52$:

$$\text{Area of } \triangle ABD = \tfrac{1}{2}(5.30)(3.52) = 9.328$$

In $\triangle BCD$, $b = 5.5$, and $h = 3.2$:

$$\text{Area of } \triangle ABD = \tfrac{1}{2}(5.5)(3.2) = 8.80$$

The first area should be given to three significant digits, the least number of significant digits involved in the calculation. Similarly, the second area should be given to two significant digits. However, the area of *ABCD* should be no more precise than the least precise measurement of the areas that are added. Since the least precise measurement is 8.8, given to the nearest tenth, the area of *ABCD* should also be written to the nearest tenth:

$$\begin{aligned}\text{Area of } ABCD &= \text{Area of } \triangle ABD + \text{Area of } \triangle BCD \\ &= 9.328 + 8.80 \\ &= 18.128 \\ &= 18.1\end{aligned}$$

Answer 18.1 square centimeters

Note: When working with significant digits involving multiple operations, make sure to keep at least one extra digit for intermediate calculations to avoid *round-off error*.

Alternatively, when working with a calculator, you can round at the end of the entire calculation.

EXERCISES

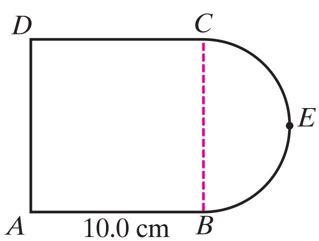
Writing About Mathematics

1. *ABCD* is a trapezoid with $\overline{AB} \parallel \overline{CD}$. The area of $\triangle ABD$ is 35 square units. What is the area of $\triangle ABC$? Explain your answer.

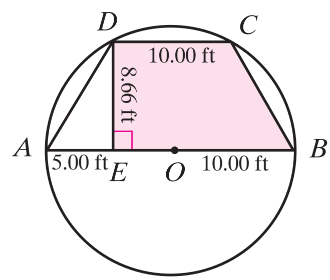
2. $ABCD$ is a quadrilateral. The area of $\triangle ABD$ is 57 square inches, of $\triangle BDC$ is 62 square inches, and of $\triangle ABC$ is 43 square inches. What is the area of $\triangle ADC$? Explain your answer.

Developing Skills

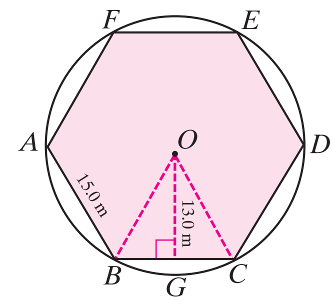
In 3–10, find each measure to the correct number of significant digits.



Ex. 3



Ex. 4

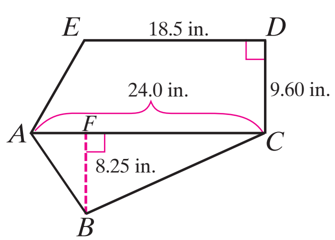


Ex. 5

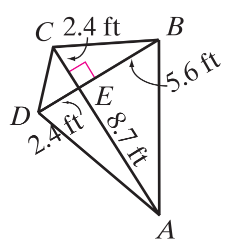
3. $ABCD$ is a square and arc $\widehat{BEC}$ is a semicircle. If $AB = 10.0$ cm, find the area of the figure.

4. $ABCD$ is a trapezoid. The vertices of the trapezoid are on a circle whose center is at O. $\overline{DE} \perp \overline{AB}$, $OB = DC = 10.00$ ft, $DE = 8.66$ ft, and $AE = 5.00$ ft. Find the area of $EBCD$.

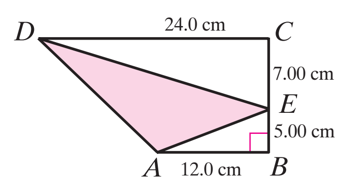
5. $ABCDEF$ is a regular hexagon. The vertices of the hexagon are on a circle whose center is at O. $\overline{OG} \perp \overline{BC}$, $AB = 15.0$ m and $OG = 13.0$ m. Find the area of $ABCDEF$.



Ex. 6



Ex. 7

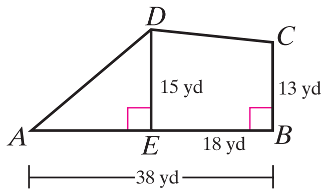


Ex. 8

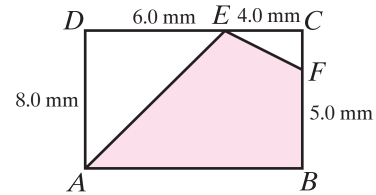
6. $ABCDE$ is a pentagon. $\overline{DC} \perp \overline{ED}$, $\overline{ED} \parallel \overline{AC}$, and $\overline{BF} \perp \overline{AC}$. If $ED = 18.5$ in., $DC = 9.60$ in., $AC = 24.0$ in., and $BF = 8.25$ in., find the area of $ABCDE$.

7. $ABCD$ is a quadrilateral. The diagonals of the quadrilateral are perpendicular to each other at E. If $AE = 8.7$ ft, $BE = 5.6$ ft, $CE = 2.4$ ft, and $DE = 2.4$ ft, find the area of $ABCD$.

8. $ABCD$ is a trapezoid with $\overline{BC} \perp \overline{AB}$ and E a point on $\overline{BC}$. $AB = 12.0$ cm, $DC = 24.0$ cm, $BE = 5.00$ cm, and $EC = 7.00$ cm. Find the area of triangle AED.



Ex. 9



Ex. 10

9. $ABCD$ is a quadrilateral with $\overline{BC} \perp \overline{AB}$ and $\overline{DE} \perp \overline{AB}$. If $AB = 38$ yd, $EB = 18$ yd, $BC = 13$ yd, and $DE = 15$ yd. Find the area of $ABCD$.

10. $ABCD$ is a rectangle. $AD = 8$ mm, $DE = 6$ mm, $EC = 4$ mm, and $BF = 5$ mm. Find the area of $ABFE$.

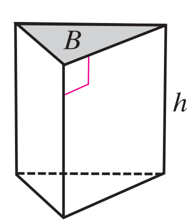
Applying Skills

11. Mason has a square of fabric that measures one yard on each side. He makes a straight cut from the center of one edge to the center of an adjacent edge. He now has one piece that is an isosceles triangle and another piece that is a pentagon. Find the area of each of these pieces in square inches.

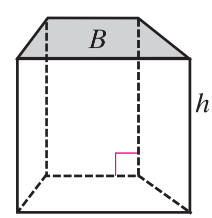
12. A park is in the shape of isosceles trapezoid $ABCD$. The bases of the trapezoid, $\overline{AB}$ and $\overline{CD}$ measure 20.0 meters and 5.00 meters respectively. The measure of each base angle at $\overline{AB}$ is 60°, and the height of the trapezoid is 13.0 meters. A straight path from A to E, a point on $\overline{BC}$, makes an angle of 30° with $\overline{AB}$ and $BE = 10.0$ meters. The path separates the trapezoid into two regions; a quadrilateral planted with grass and a triangle planted with shrubs and trees. Find the area of the region planted with grass to the nearest meter.

7-7 SURFACE AREAS OF SOLIDS

A **right prism** is a solid with congruent bases and with a height that is perpendicular to these bases. Some examples of right prisms, as seen in the diagrams, include solids whose bases are triangles, trapezoids, and rectangles. The prism is named for the shape of its base. The two bases may be any polygons, as long as they have the same size and shape. The remaining sides, or **faces**, are rectangles.



Triangular right prism

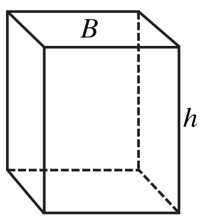


Trapezoidal right prism

The **surface area** of a solid is the sum of the areas of the surfaces that make up the solid. The surface area of a right prism is the sum of the areas of the bases and the faces of the solid. The number of faces of a prism is equal to the number of sides of a base.

- If the base of the prism is a triangle, it has 2 bases + 3 faces or 5 surfaces.
- If the base of the prism is a quadrilateral, it has 2 bases + 4 faces or 6 surfaces.
- If the base of the prism has n sides, it has 2 bases + n faces or $2 + n$ surfaces.

Two or more of the faces are congruent if and only if two or more of the sides of a base are equal in length. For example, if the bases are isosceles triangles, two of the faces will be congruent rectangles and if the bases are equilateral triangles, all three of the faces will be congruent rectangles. If the bases are squares, then the four faces will be congruent rectangles.



Rectangular solid (a prism)

The most common right prism is a **rectangular solid** that has rectangles as bases and as faces. Any two surfaces that have no edge in common can be the bases and the other four surfaces are the faces. If the dimensions of the rectangular solid are 3 by 5 by 4, then there are two rectangles that are 3 by 5, two that are 5 by 4 and two that are 3 by 4. The surface area of the rectangular solid is:

$$2(3)(5) + 2(5)(4) + 2(3)(4) = 30 + 40 + 24 = 94 \text{ square units}$$

In general, when the dimensions of a rectangular solid are represented by l, w, and h, then the formula for the surface area is:

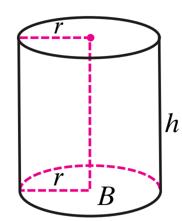
Surface Area of a Rectangular Solid = $2lw + 2lh + 2wh$

If the rectangular solid has six squares as bases and sides, the solid is a cube. If s represents the length of each side, then $l = s$, $w = s$, and $h = s$. If we substitute in the formula for the area of a rectangular solid, then:

$$S = 2lw + 2lh + 2wh = 2s(s) + 2s(s) + 2s(s) = 2s^2 + 2s^2 + 2s^2 = 6s^2$$

Surface Area of a Cube = $6s^2$

A **right circular cylinder** is a solid with two bases that are circles of the same size, and with a height that is perpendicular to these bases.



Right circular cylinder

Fruits and vegetables are often purchased in "cans" that are in the shape of a right circular cylinder. The label on the can corresponds to the surface area of the curved portion of the cylinder. That label is a rectangle whose length is the circumference of the can, $2\pi r$, and whose width is the height of the can, h. Therefore, the area of the curved portion of the cylinder is $2\pi rh$ and the area of each base is πr^2.

Surface Area of a Cylinder = 2 × the area of a base + the area of the curved portion

Surface Area of a Cylinder = $2\pi r^2 + 2\pi rh$

EXAMPLE 1

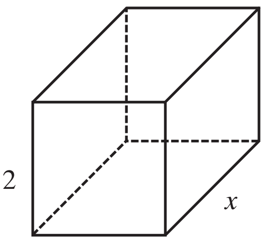
A rectangular solid has a square base. The height of the solid is 2 less than twice the length of one side of the square. The height of the solid is 22.2 centimeters. Find the surface area of the figure to the correct number of significant digits.

Solution

Let x = the length of a side of the base.

$2x - 2$ = the height of the figure.

$$2x - 2 = 22.2$$
$$2x = 24.2$$
$$x = 12.1$$



The surface area consists of two square bases with sides that measure 12.1 centimeters and four rectangular faces with width 12.1 and length 22.2.

Surface area of the solid = surface area of the bases + surface area of the faces

$$= 2(12.1)^2 + 4(12.1)(22.2)$$
$$= 292.82 + 1{,}074.48$$
$$= 1{,}367.3$$
$$= 1{,}370$$

The areas of the bases should be given to three significant digits, the least number of significant digits involved in the calculation. Similarly, the areas of the faces should be given to three significant digits. However, the surface area of the solid should be no more precise than the least precise measurement of the areas that are added. Since the least precise measurement is 1,074.48, given to the nearest ten, the surface area of the solid should also be written to the nearest ten: 1,370 square centimeters.

Answer Surface area = 1,370 cm^2

Note: Recall that when working with significant digits involving multiple operations, you should keep at least one extra digit for intermediate calculations *or* wait until the end of the entire calculation to round properly.

EXERCISES

Writing About Mathematics

1. A right prism has bases that are regular hexagons. The measure of each of the six sides of the hexagon is represented by a and the height of the solid by $2a$.

a. How many surfaces make up the solid?

b. Describe the shape of each face

c. Express the dimensions and the area of each face in terms of a.

2. A regular hexagon can be divided into six equilateral triangles. If the length of a side of an equilateral triangle is a, the height is $\frac{\sqrt{3}}{2}a$. For the rectangular solid described in exercise 1:

a. Express the area of each base in terms of a.

b. Express the surface area in terms of a.

Developing Skills

In 3–9, find the surface area of each rectangular prism or cylinder to the nearest *tenth of a square unit*.

3. Bases are circles with a radius of 18 inches. The height of the cylinder is 48 inches.

4. Bases are squares with sides that measure 27 inches. The height is 12 inches.

5. Bases are rectangles with dimensions of 8 feet by 12 feet. The height is 3 feet.

6. Bases are isosceles trapezoids with parallel sides that measure 15 centimeters and 25 centimeters. The distance between the parallel sides is 12 centimeters and the length of each of the equal sides is 13 centimeters. The height of the prism is 20 centimeters.

7. Bases are isosceles right triangles with legs that measure 5 centimeters. The height is 7 centimeters.

8. Bases are circles with a diameter of 42 millimeters. The height is 3.4 centimeters.

9. Bases are circles each with an area of 314.16 square feet. The height is 15 feet.

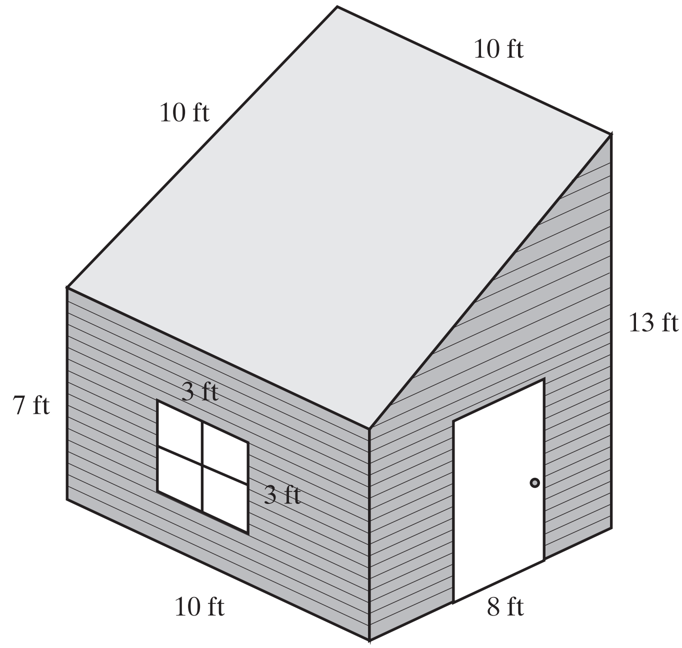
Applying Skills

10. Agatha is using scraps of wallpaper to cover a box that is a rectangular solid whose base measures 8 inches by 5 inches and whose height is 3 inches. The box is open at the top. How many square inches of wallpaper does she need to cover the outside of the box?

11. Agatha wants to make a cardboard lid for the box described in exercise 10. Her lid will be a rectangular solid that is open at the top, with a base that is slightly larger than that of the box. She makes the base of the lid 8.1 inches by 5.1 inches with a height of 2.0 inches. To the nearest tenth of a square inch, how much wallpaper does she need to cover the outside of the lid?

12. Sandi wants to make a pillow in the shape of a right circular cylinder. The diameter of the circular ends is 10.0 inches and the length of the pillow (the height of the cylinder) is 16.0 inches. Find the number of square inches of fabric Sandi needs to make the pillow to the correct number of significant digits.

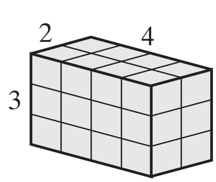
13. Mr. Breiner made a tree house for his son. The front and back walls of the house are trapezoids to allow for a slanted roof. The floor, roof, and remaining two sides are rectangles. The tree house is a rectangular solid. The front and back walls are the bases of this solid. The dimensions of the floor are 8 feet by 10 feet and the roof is 10 feet by 10 feet. The height of one side wall is 7 feet and the height of the other is 13 feet. The two side walls each contain a window measuring 3 feet by 3 feet. Mr. Breiner is going to buy paint for the floor and the exterior of the house, including the roof, walls, and the door. How many square feet must the paint cover, excluding the windows?



7-8 VOLUMES OF SOLIDS

The **volume** of a solid is the number of unit cubes (or cubic units) that it contains. Both the volume V of a right prism and the volume V of a right circular cylinder can be found by multiplying the area B of the base by the height h. This formula is written as

$$V = Bh$$



$V = Bh$
$V = (lw)h$
$= (4 \cdot 2) \cdot 3$
$= 8 \cdot 3$
$= 24 \text{ cm}^3$

To find the volume of a solid, all lengths *must* be expressed in the same unit of measure. The volume is then expressed in cubic units of this length.

For example, in the rectangular solid or right prism shown at left, the base is a rectangle, 4 centimeters by 2 centimeters. The area, B, of this base is lw. Therefore, $B = 4(2) = 8 \text{ cm}^2$. Then, using a height h of 3 centimeters, the volume V of the rectangular solid $= Bh = (8)(3) = 24 \text{ cm}^3$.

For a rectangular solid, note that the volume formula can be written in two ways:

$$V = Bh \qquad \text{or} \qquad V = lwh$$

To understand volume, count the cubes in the diagram on the previous page. There are 3 layers, each containing 8 cubes, for a total of 24 cubes. Note that 3 corresponds to the height, h, that 8 corresponds to the area of the base, B, and that 24 corresponds to the volume V in cubic units.

A cube that measures 1 foot on each side represents 1 cubic foot. Each face of the cube is 1 square foot. Since each foot can be divided into 12 inches, the area of each face is 12×12 or 144 square inches and the volume of the cube is $12 \times 12 \times 12$ or 1,728 cubic inches.

1 square foot = 144 square inches

1 cubic foot = 1,728 cubic inches

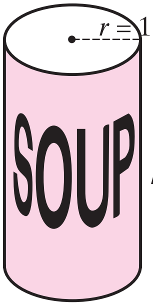
A cube that measures 1 meter on each side represents 1 cubic meter. Each face of the cube is 1 square meter. Since each meter can be divided into 100 centimeters, the area of each face is 100×100 or 10,000 square centimeters and the volume of the cube is $100 \times 100 \times 100$ or 1,000,000 cubic centimeters.

1 square meter = 10,000 square centimeters

1 cubic meter = 1,000,000 cubic centimeters

EXAMPLE 1

A cylindrical can of soup has a radius of 1.5 inches and a height of 5 inches. Find the volume of this can:



a. in terms of π.

b. to the nearest cubic inch.

Solution This can is a right circular cylinder. Use the formula $V = Bh$. Since the base is a circle whose area equals πr^2, the area B of the base can be replaced by πr^2.

a. $V = Bh = (\pi r^2)h = \pi(1.5)^2(5) = 11.25\pi$

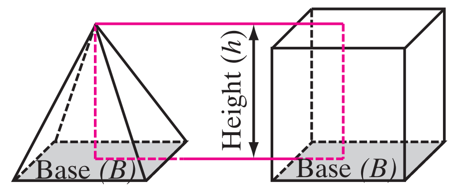
b. When we use a calculator to evaluate 11.25π, the calculator gives 35.34291735 as a rational approximation. This answer rounded to the nearest integer is 35.

Answers **a.** 11.25π cu in. **b.** 35 cu in.

A **pyramid** is a solid figure with a base that is a polygon and triangular faces or sides that meet at a common point. The formula for the volume V of a pyramid is:

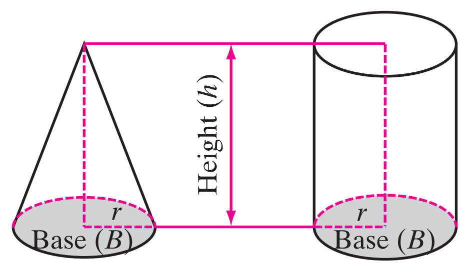
$$V = \frac{1}{3}Bh$$

In the pyramid and prism shown here, the bases are the same in size and shape, and the heights are equal in measure. If the pyramid could be filled with water and that water poured into the prism, exactly three pyramids of water would be needed to fill the prism. In other words, the volume of a pyramid is one-third the volume of a right prism with the same base and same height as those of the pyramid.



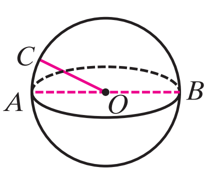
Pyramid = $\frac{1}{3}$ prism

The volume of a **cone** is one-third of the volume of a right circular cylinder, when both the cone and the cylinder have circular bases and heights that are equal in measure. The formula for the volume of the cone is



$$V = \frac{1}{3}Bh \qquad \text{or} \qquad V = \frac{1}{3}\pi r^2 h$$

As in the case of the pyramid and prism, if the cone could be filled with water and that water poured into the right circular cylinder, exactly three cones of water would be needed to fill the cylinder.



Sphere

A **sphere** is a *solid* figure whose points are all equally distant from one fixed point in space called its **center.** A sphere is *not* a circle, which is drawn on a flat surface or plane but similar terminology is used for a sphere. A line segment, such as $\overline{OC}$, that joins the center O to any point on the sphere is called a **radius** of the sphere. A line segment, such as $\overline{AB}$, that joins two points of the sphere and passes through its center is called a **diameter** of the sphere.

The volume of a sphere with radius r is found by using the formula:

$$V = \frac{4}{3}\pi r^3$$

EXAMPLE 2

An ice cream cone that has a diameter of 6.4 centimeters at its top and a height of 12.2 centimeters is filled with ice cream and topped with a scoop of ice cream that is approximately in the shape of half of a sphere. Using the correct number of significant digits, how many cubic centimeters of ice cream are needed?

Solution The amount of ice cream needed is approximately the volume of the cone plus one-half the volume of a sphere with a radius equal to the radius of the top of the cone.

$$\begin{aligned}\text{Volume of ice cream} &= \tfrac{1}{3}\pi r^2 h + \tfrac{1}{2}\left(\tfrac{4}{3}\pi r^3\right)\\ &= \tfrac{1}{3}\pi(3.2)^2(12.2) + \tfrac{1}{2}\left(\tfrac{4}{3}\right)\pi(3.2)^3\\ &= 130.824\ldots + 68.629\ldots\\ &= 199.453\ldots\end{aligned}$$

The least precise volume is the volume of the cone, given to the nearest ten. Therefore, the answer should be rounded to the nearest ten. Note that in the answer, the zero in the tens place is significant. The zero in the ones place is not significant.

Answer The volume of ice cream is 200 cubic centimeters.

Error in Geometric Calculations

When a linear measure is used to find area or volume, any error in the linear value will be increased in the higher-dimension calculations.

EXAMPLE 3

The length of a side of a cube that is actually 10.0 centimeters is measured to be 10.5 centimeters. Find the percent error in:

a. the linear measure.

b. the surface area.

c. the volume.

d. compare the results in **a**–**c** above.

Solution **a.** The true length is 10.0 centimeters. The measured value is 10.5 centimeters.

$$\begin{aligned}\text{Percent error} &= \frac{|\text{measured value} - \text{true value}|}{\text{true value}} \times 100\%\\ &= \frac{|10.5 - 10.0|}{10.0} \times 100\% = \frac{0.5}{10.0} \times 100\%\\ &= 0.05 \times 100\% = 5\% \quad \textit{Answer}\end{aligned}$$

b. For a cube, $S = 6s^2$.

Using the measured value: $S = 6(10.5)^2 = 661.5$

Using the true value: $S = 6(10.0)^2 = 600.0$

$$\text{Percent error} = \frac{|661.5 - 600.0|}{600.0} \times 100\% = \frac{61.5}{600.0} \times 100\%$$
$$= 0.1025 \times 100\% = 10.25\% \quad \textit{Answer}$$

c. For a cube, $V = s^3$.

Using the measured value: $V = (10.5)^3 = 1{,}157.625$

Using the true value: $V = (10.0)^3 = 1{,}000$

$$\text{Percent error} = \frac{|1{,}157.625 - 1{,}000|}{1{,}000} \times 100\% = \frac{157.625}{1{,}000} \times 100\%$$
$$= 0.157625 \times 100\% \approx 15.76\% \quad \textit{Answer}$$

d. The error in the linear measure is 5%. The error increases to 10.25% when the surface area is calculated. The error increased even more to 15.76% when the volume is calculated. *Answer*

EXERCISES

Writing About Mathematics

1. A chef purchases thin squares of dough that she uses for the top crust of pies. The squares measure 8 inches on each side. From each square of dough she cuts either one circle with a diameter of 8 inches or four circles, each with a diameter of 4 inches.

a. Compare the amount of dough left over when she makes one 8-inch circle with that left over when she makes four 4-inch circles.

b. The chef uses the dough to form the top crust of pot pies that are 1.5 inches deep. Each pie is approximately a right circular cylinder. Compare the volume of one 8-inch pie to that of one 4-inch pie.

2. Tennis balls can be purchased in a cylindrical can in which three balls are stacked one above the other. How does the radius of each ball compare with the height of the can in which they are packaged?

Developing Skills

In 3–6, use the formula $V = lwh$ and the given dimensions to find the volume of each rectangular solid using the correct number of significant digits.

3. $l = 5.0$ ft, $w = 4.0$ ft, $h = 7.0$ ft

4. $l = 8.5$ cm, $w = 4.2$ cm, $h = 6.0$ cm

5. $l = 2\frac{1}{2}$ m, $w = 3\frac{1}{4}$ cm, $h = 85$ cm

6. $l = 7.25$ in., $w = 6.40$ in., $h = 0.25$ ft

7. Find the volume of a cube if each edge measures $8\frac{3}{5}$ centimeters.

8. The measure of each edge of a cube is represented by $(2y - 3)$ centimeters. Find the volume of the cube when $y = 7.25$.

9. The base of a right prism is a triangle. One side of the triangular base measures 8 centimeters and the altitude to that side measures 6 centimeters. The height h of the prism measures 35 millimeters.

a. Find the area of the triangular base of the prism.

b. Find the volume of the prism.

10. The base of a right prism is a trapezoid. This trapezoid has bases that measure 6 feet and 10 feet and an altitude that measures 4 feet. The height h of the prism is 2 yards.

a. Find the area of the trapezoidal base of the prism.

b. Find the volume of the prism.

11. The base of a pyramid has an area B of 48 square millimeters and a height h of 13 millimeters. What is the volume of the pyramid?

12. The height of a pyramid is 4 inches, and the base is a rectangle 6 inches long and $3\frac{1}{2}$ inches wide. What is the volume of the pyramid?

13. A right circular cylinder has a base with a radius of 24.1 centimeters and a height of 17.3 centimeters.

a. Express the volume of the cylinder in terms of π.

b. Find the volume of the cylinder to the nearest hundred cubic centimeters.

14. A right circular cylinder has a base with a diameter of 25 meters and a height of 15 meters.

a. Express the volume of the cylinder in terms of π.

b. Find the volume of the cylinder to the nearest ten cubic meters.

15. The base of a cone has a radius of 7 inches. The height of the cone is 5 inches.

a. Find the volume of the cone in terms of π.

b. Find the volume of the cone to the nearest cubic inch.

16. The base of a cone has a radius of 7 millimeters. The height of the cone is 2 centimeters.

a. Find the volume of the cone in terms of π.

b. Express the volume of the cone to the nearest cubic centimeter.

17. A sphere has a radius of 12.5 centimeters.

a. Find the volume of the sphere in terms of π.

b. Express the volume of the sphere to the nearest cubic centimeter.

18. A sphere has a diameter of 3 feet.

a. Find the volume of the sphere in terms of π.

b. Express the volume of the sphere to the nearest cubic foot.

19. The side of a cube that is actually 8 inches is measured to be 7.6 inches. Find, to the nearest tenth of a percent, the percent error in:

a. the length of the side.

b. the surface area.

c. the volume.

Applying Skills

20. An official handball has a diameter of 4.8 centimeters. Find its volume:

a. to the nearest cubic centimeter **b.** to the nearest cubic inch.

21. A tank in the form of a right circular cylinder is used for storing water. It has a diameter of 12 feet and a height of 14 feet. How many gallons of water will it hold? (1 cubic foot contains 7.5 gallons.)

22. Four pieces of cardboard that are 8.0 inches by 12 inches and two pieces that are 12 inches by 12 inches are used to form a rectangular solid.

a. What is the surface area of the rectangular solid formed by the six pieces of cardboard?

b. What is the volume of the rectangular solid in cubic inches?

c. What is the volume of the rectangular solid in cubic feet?

23. A can of soda is almost in the shape of a cylinder with a diameter of 6.4 centimeters and a height of 12.3 centimeters.

a. What is the volume of the can?

b. If there are 1,000 cubic centimeters in a liter, find how many liters of soda the can holds.

In 24–26, express each answer to the correct number of significant digits.

24. Cynthia used a shipping carton that is a rectangular solid measuring 12.0 inches by 15.0 inches by 3.20 inches. What is the volume of the carton?

25. The highway department stores sand in piles that are approximately the shape of a cone. What is the volume of a pile of sand if the diameter of the base is 7.0 yards and the height of the pile is 8.0 yards?

26. The largest pyramid in the world was built around 2500 B.C. by Khufu, or Cheops, a king of ancient Egypt. The pyramid had a square base $23\dot{0}$ meters (756 feet) on each side, and a height of 147 meters (482 feet). (The length of a side of the base is given to the nearest meter. The zero in the ones place is significant.) Find the volume of Cheops' pyramid using:
a. cubic meters **b.** cubic feet

CHAPTER SUMMARY

Point, **line**, and **plane** are undefined terms that are used to define other terms. A **line segment** is a part of a line consisting of two points on the line, called **endpoints**, and all of the points on the line between the endpoints.

A **ray** is a part of a line that consists of a point on the line and all of the points on one side of that point. An **angle** is the union of two rays with a common endpoint.

Two angles are **complementary** if the sum of their measures is 90°. If the measure of an angle is $x°$, the measure of its complement is $(90 - x)°$.

Two angles are **supplementary** if the sum of their measures is 180°. If the measure of an angle is $x°$, the measure of its supplement is $(180 - x)°$.

A **linear pair of angles** are adjacent angles that are supplementary. Two angles are **vertical angles** if the sides of one are opposite rays of the sides of the other. Vertical angles are **congruent**.

If two parallel lines are cut by a transversal, then:

- The alternate interior angles are congruent.
- The alternate exterior angles are congruent.
- The corresponding angles are congruent.
- Interior angles on the same side of the transversal are supplementary.

The sum of the measures of the angles of a triangle is 180°.

The base angles of an isosceles triangle are congruent.

An equilateral triangle is equiangular.

The sum of the measures of the angles of a quadrilateral is 360°. The sum of the measures of the angles of any polygon with n sides is $180(n - 2)$.

If the measures of the bases of a trapezoid are b_1 and b_2 and the measure of the altitude is h, then the formula for the area of the trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$.

Formulas for Surface Area
Rectangular solid: $S = 2lw + 2lh + 2wh$
Cube: $S = 6s^2$
Cylinder: $S = 2\pi r^2 + 2\pi rh$

Formulas for Volume
Any right prism: $V = Bh$
Rectangular solid: $V = lwh$
Right circular cylinder: $V = \pi r^2 h$
Pyramid: $V = \frac{1}{3}Bh$
Cone: $V = \frac{1}{3}\pi r^2 h$
Cube: $V = s^3$
Sphere: $V = \frac{4}{3}\pi r^3$

VOCABULARY

7-1 Undefined term • Point • Line • Straight line • Plane • Axiom (postulate) • Line segment (segment) • Endpoints of a segment • Measure of a line segment (length of a line segment) • Half-line • Ray • Endpoint of a ray • Opposite rays • Angle • Vertex of an angle • Sides an angle • Degree • Right angle • Acute angle • Obtuse angle • Straight angle • Perpendicular

7-2 Adjacent angles • Complementary angles • Complement • Supplementary angles • Linear pair of angles • Vertical angles • Congruent angles • Theorem

7-3 Parallel lines • Transversal • Interior angles • Alternate interior angles • Exterior angles • Alternate exterior angles • Interior angles on the same side of the transversal • Corresponding angles

7-4 Polygon • Sides of a polygon • Vertices • Triangle • Acute triangle • Equiangular triangle • Right triangle • Obtuse triangle • Legs of a right triangle • Hypotenuse • Scalene triangle • Isosceles triangle • Equilateral triangle • Legs of an isosceles triangle • Base of an isosceles triangle • Congruent line segments • Vertex angle of an isosceles triangle • Base angles of an isosceles triangle • Construction • Compass • Straightedge • Perpendicular bisector • Angle bisector

7-5 Quadrilateral • Consecutive angles • Opposite angles • Trapezoid • Bases of a trapezoid • Parallelogram • Rectangle • Rhombus • Square • Isosceles trapezoid

7-7 Right prism • Face • Surface area • Rectangular solid • Right circular cylinder

7-8 Volume • Pyramid • Cone • Sphere • Center of a sphere • Radius of a sphere • Diameter of a sphere

REVIEW EXERCISES

1. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, $m\angle AEC = x + 10$, and $m\angle DEB = 2x - 30$, find $m\angle AEC$.

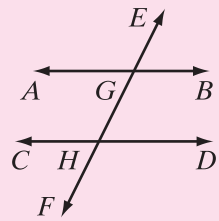
2. If two angles of a triangle are complementary, what is the measure of the third angle?

3. The measure of the complement of an angle is 20° less than the measure of the angle. Find the number of degrees in the angle.

4. If each base angle of an isosceles triangle measures 55°, find the measure of the vertex angle of the triangle.

5. The measures of the angles of a triangle are consecutive even integers. What are the measures of the angles?

In 6–8, $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, and these lines are cut by transversal $\overleftrightarrow{EF}$ at points G and H, respectively.



6. If $m\angle AGH = 73$, find $m\angle GHD$.

7. If $m\angle EGB = 70$ and $m\angle GHD = 3x - 2$, find x.

8. If $m\angle HGB = 2x + 10$ and $m\angle GHD = x + 20$, find $m\angle GHD$.

9. In $\triangle ART$, $m\angle A = y + 10$, $m\angle R = 2y$, and $m\angle T = 2y - 30$.

a. Find the measure of each of the three angles.

b. Is $\triangle ART$ acute, right, or obtuse?

c. Is $\triangle ART$ scalene, isosceles but not equilateral or equiangular?

10. The measure of each base angle of an isosceles triangle is 15° more than the measure of the vertex angle. Find the measure of each angle.

11. The measure of an angle is 20° less than 3 times the measure of its supplement. What is the measure of the angle and its supplement?

12. In $\triangle ABC$, the measure of $\angle B$ is $\frac{3}{2}$ the measure of $\angle A$, and the measure of $\angle C$ is $\frac{5}{2}$ the measure of $\angle A$. What are the measures of the three angles?

13. The measure of one angle is 3 times that of another angle, and the sum of these measures is 120°. What are the measures of the angles?

14. The measure of the smaller of two supplementary angles is $\frac{4}{5}$ of the measure of the larger. What are the measures of the angles?

15. The vertices of a trapezoid are $A(-3, -1)$, $B(7, -1)$, $C(5, 5)$, and $D(-7, 5)$.

a. Draw $ABCD$ on graph paper.

b. E is the point on $\overline{CD}$ such that $\overline{CD} \perp \overline{AE}$. What are the coordinates of E?

c. Find AB, CD, and AE.

d. Find the area of trapezoid $ABCD$.

16. In parallelogram $ABCD$, $AB = 3x + 4$, $BC = 2x + 5$, and $CD = x + 18$. Find the measure of each side of the parallelogram.

17. A flowerpot in the shape of a right circular cylinder has a height of 4.5 inches. The diameter of the base of the pot is 4.1 inches. Find the volume of the pot to the nearest tenth.

18. Natali makes a mat in the shape of an octagon (an eight-sided polygon) by cutting four isosceles right triangles of equal size from the corners of a 9 by 15 inch rectangle. If the measure of each of the equal sides of the triangles is 2 inches, what is the area of the octagonal mat?

19. Marvin measures a block of wood and records the dimensions as 5.0 centimeters by 3.4 centimeters by 4.25 centimeters. He places the block of wood in a beaker that contains 245 milliliters of water. With the block completely submerged, the water level rises to 317 milliliters.

a. Use the dimensions of the block to find the volume of the block of wood.

b. What is the volume of the block of wood based on the change in the water level?

c. Marvin knows that 1 ml = 1 cm^3. Can the answers to parts **a** and **b** both be correct? Explain your answer.

20. A watering trough for cattle is in the shape of a prism whose ends are the bases of the prism and whose length is the height of the prism. The ends of the trough are trapezoids with parallel edges 31 centimeters and 48 centimeters long and a height of 25 centimeters. The length of the trough is 496 centimeters.

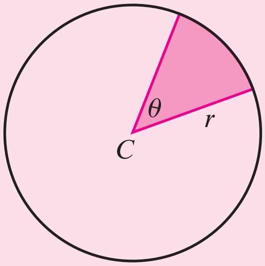
a. Find the surface area of one end of the trough.

b. If the trough is filled to capacity, how many cubic centimeters of water does it hold?

c. How many liters of water does the trough hold? (1 liter = 1,000 cm^3)

Exploration

A **sector** of a circle is a fractional part of the interior of a circle, determined by an angle whose vertex is at the center of the circle (a **central angle**). The area of a sector depends on the measure of its central angle. For example,



The shaded region represents a sector with central angle θ.

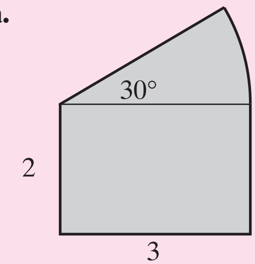
- If the central angle equals 90°, then the area of the sector is one-fourth the area of the circle, or $\frac{90}{360}\pi r^2$.
- If the central angle equals 180°, then the area of the sector is one-half the area of the circle, or $\frac{180}{360}\pi r^2$.
- If the central angle equals 270°, then the area of the sector is three-fourths the area of the circle, or $\frac{270}{360}\pi r^2$.

In general, if the measure of the central angle is θ (theta), then the area of the sector is

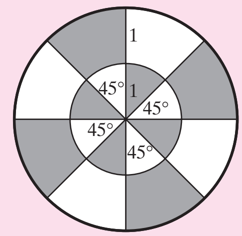
$$\textbf{Area of a sector} = \frac{\theta}{360}\pi r^2$$

For this Exploration, use your knowledge of Geometry and the formula for the area of a sector to find the areas of the shaded regions. Express your answers in terms of π. Assume that all the arcs that are drawn are circular.

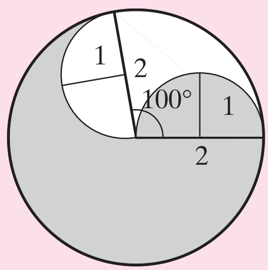
a.



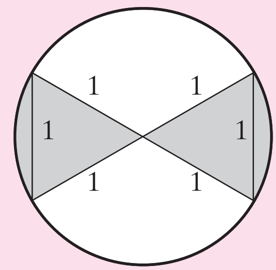
b.



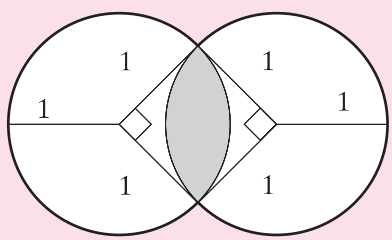
c.



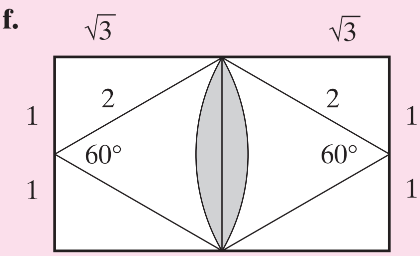
d.



e.



f.



CUMULATIVE REVIEW CHAPTERS 1–7

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following inequalities is true?

(1) $-|-4| < -(-3) < |-5|$
(2) $-|-4| < |-5| < -(-3)$
(3) $-(-3) < -|-4| < |-5|$
(4) $|-5| < -|-4| < -(-3)$

2. Which of the following is an example of the use of the associative property?

(1) $2(x + 5) = 2(5 + x)$
(2) $2(x + 5) = 2x + 2(5)$
(3) $2 + (x + 5) = 2 + (5 + x)$
(4) $2 + (5 + x) = (2 + 5) + x$

3. Which of the following is not a rational number?

(1) $\sqrt{0.09}$ (2) $\sqrt{0.9}$ (3) 2^{-3} (4) $0.\overline{15}$

4. The product $(a^2b)(a^3b)$ is equivalent to

(1) a^6b (2) a^5b (3) a^5b^2 (4) a^6b^2

5. The solution set of $\frac{2}{3}x + 7 = \frac{1}{6}x - 5$ is

(1) $\{24\}$ (2) $\{-24\}$ (3) $\{6\}$ (4) $\{-6\}$

6. The sum of $b^2 - 7$ and $b^2 + 3b$ is

(1) $b^4 - 4b$ (2) $b^4 + 3b - 7$ (3) $2b^2 - 4b$ (4) $2b^2 + 3b - 7$

7. Two angles are supplementary. If the measure of one angle is 85°, the measure of the other is

(1) 5° (2) 85° (3) 95° (4) 180°

8. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$. If $m\angle A$ is 75°, then $m\angle D$ is

(1) 15° (2) 75° (3) 105° (4) 165°

9. The graph at the right is the solution set of

(1) $-2 \leq x < 3$
(2) $-2 < x < 3$
(3) $-2 < x \leq 3$
(4) $(-2 > x)$ or $(x \geq 3)$

10. When $a = -1.5$, $-3a^2 - a$ equals

(1) -5.25 (2) 5.25 (3) 18.75 (4) 21.75

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The area of a rectangle is $(x^2 + 6x + 8)$ square inches and its length is $(x + 4)$ inches. Express the width of the rectangle in terms of x.

12. Solve the following equation for x: $\frac{7}{x - 2} = \frac{3}{x + 2}$.

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Mr. Popowich mailed two packages. The larger package weighed 12 ounces more than the smaller. If the total weight of the packages was 17 pounds, how much did each package weigh?

14. a. Solve for x in terms of a and b: $ax + 3b = 7$.

b. Find, to the nearest hundredth, the value of x when $a = \sqrt{3}$ and $b = \sqrt{5}$.

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Calvin traveled 600 miles, averaging 40 miles per hour for part of the trip and 60 miles per hour for the remainder of the trip. The entire trip took 11 hours. How long did Calvin travel at each rate?

16. A box used for shipping is in the shape of a rectangular prism. The bases are right triangles. The lengths of the sides of the bases are 9.0, 12, and 15 feet. The height of the prism is 4.5 feet.

a. Find the surface area of the prism. Express the answer using the correct number of significant digits.

b. Find the volume of the prism. Express the answer using the correct number of significant digits.